

# Mixed boundary value problems for the Navier-Stokes system in polyhedral domains

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## Abstract

Mixed boundary value problems for the Navier-Stokes system in a polyhedral domain are considered. Different boundary conditions (in particular, Dirichlet, Neumann, slip conditions) are prescribed on the faces of the polyhedron. The authors obtain regularity results for weak solutions in weighted (and non-weighted)  $L_p$  Sobolev and Hölder spaces.

Keywords: Navier-Stokes system, nonsmooth domains

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## 0 Introduction

Steady-state flows of incompressible viscous Newtonian fluids are modeled by the Navier-Stokes equations

$$-\nu \Delta u + (u \cdot \nabla) u + \nabla p = f, \quad \nabla \cdot u = g \quad (0.1)$$

for the velocity  $u$  and the pressure  $p$ . For this system, one can consider different boundary conditions. For example on solid walls, we have the Dirichlet condition  $u = 0$ . On other parts of the boundary (an artificial boundary such as the exit of a channel, or a free surface) a no-friction condition (Neumann condition)  $2\nu \varepsilon(u) n - pn = 0$  may be useful. Here  $\varepsilon(u)$  denotes the matrix with the components  $\frac{1}{2}(\partial_{x_i} u_j + \partial_{x_j} u_i)$ , and  $n$  is the outward normal. Note that the Neumann problem naturally appears in the theory of hydrodynamic potentials (see [12]). It is also of interest to consider boundary conditions containing components of the velocity and of the friction. Frequently used combinations are the normal component of the velocity and the tangential component of the friction (slip condition for uncovered fluid surfaces) or the tangential component of the velocity and the normal component of the friction (condition for in/out-stream surfaces).

In the present paper, we consider mixed boundary value problems for the system (0.1) in a three-dimensional domain  $\mathcal{G}$  of polyhedral type, where components of the velocity and/or the friction are given on the boundary. To be more precise, we have one of the following boundary conditions on each face  $\Gamma_j$ :

- (i)  $u = h$ ,
- (ii)  $u_\tau = h, \quad -p + 2\varepsilon_{n,n}(u) = \phi$ ,
- (iii)  $u_n = h, \quad \varepsilon_{n,\tau}(u) = \phi$ ,
- (iv)  $-pn + 2\varepsilon_n(u) = \phi$ ,

where  $u_n = u \cdot n$  denotes the normal and  $u_\tau = u - u_n n$  the tangential component of  $u$ ,  $\varepsilon_n(u)$  is the vector  $\varepsilon(u) n$ ,  $\varepsilon_{n,n}(u)$  is the normal component and  $\varepsilon_{n,\tau}(u)$  the tangential component of  $\varepsilon_n(u)$ .

Weak solutions, i.e. variational solutions  $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ , always exist if the data are sufficiently small. In the case when the boundary conditions (ii) and (iv) disappear, such solutions exist for arbitrary  $f$  (see the books by Ladyzhenskaya [12], Temam [25], Girault and Raviart [5]). Our goal is to prove regularity assertions for weak solutions. As is well-known, the local regularity result

$$(u, p) \in W^{l,s} \times W^{l-1,s}$$

is valid outside an arbitrarily small neighborhood of the edges and vertices if the data are sufficiently smooth. Here  $W^{l,s}$  denotes the Sobolev space of functions which belong to  $L_s$  together with all derivatives up to order  $l$ . The same result holds for the Hölder space  $C^{l,\sigma}$ . Since solutions of elliptic boundary value problems in general have singularities near singular boundary points, the result cannot be globally true in  $\mathcal{G}$  without any restrictions on  $l$  and  $s$ . Here we give a few particular regularity results which are consequences of more general theorems proved in the present paper. Suppose that the data belong to corresponding Sobolev or Hölder spaces and satisfy certain compatibility conditions on the edges. Then the following smoothness of the weak solution is guaranteed and is the best possible.

- If  $(u, p)$  is a solution of the Dirichlet problem in an arbitrary polyhedron or a solution of the Neumann problem in an arbitrary Lipschitz graph polyhedron, then

$$\begin{aligned}(u, p) &\in W^{1,3+\varepsilon}(\mathcal{G})^3 \times L_{3+\varepsilon}(\mathcal{G}), \\ (u, p) &\in W^{2,4/3+\varepsilon}(\mathcal{G})^3 \times W^{1,4/3+\varepsilon}(\mathcal{G}), \\ u &\in C^{0,\varepsilon}(\mathcal{G})^3.\end{aligned}$$

Here  $\varepsilon$  is a positive number depending on the domain  $\mathcal{G}$ .

- If  $(u, p)$  is a solution of the Dirichlet problem in a convex polyhedron, then

$$\begin{aligned}(u, p) &\in W^{1,s}(\mathcal{G})^3 \times L_s(\mathcal{G}) \quad \text{for all } s, \ 1 < s < \infty, \\ (u, p) &\in W^{2,2+\varepsilon}(\mathcal{G})^3 \times W^{1,2+\varepsilon}(\mathcal{G}), \\ (u, p) &\in C^{1,\varepsilon}(\mathcal{G})^3 \times C^{0,\varepsilon}(\mathcal{G}).\end{aligned}$$

- If  $(u, p)$  is a solution of the mixed problem in an arbitrary polyhedron with the Dirichlet and Neumann boundary condition prescribed arbitrarily on different faces, then

$$(u, p) \in W^{2,8/7+\varepsilon}(\mathcal{G})^3 \times W^{1,8/7+\varepsilon}(\mathcal{G}).$$

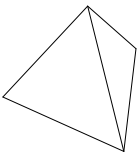
- Let  $(u, p)$  be a solution of the mixed boundary value problem with slip condition (iii) on one face  $\Gamma_1$  and Dirichlet condition on the other faces. Then

$$\begin{aligned}(u, p) &\in W^{1,3+\varepsilon}(\mathcal{G})^3 \times L_{3+\varepsilon}(\mathcal{G}) \quad \text{if } \theta < \frac{3}{2}\pi, \\ (u, p) &\in W^{2,2+\varepsilon}(\mathcal{G})^3 \times W^{1,2+\varepsilon}(\mathcal{G}) \quad \text{if } \mathcal{G} \text{ is convex and } \theta < \pi/2, \\ (u, p) &\in C^{1,\varepsilon}(\mathcal{G})^3 \times C^{0,\varepsilon}(\mathcal{G}) \quad \text{if } \mathcal{G} \text{ is convex and } \theta < \pi/2,\end{aligned}$$

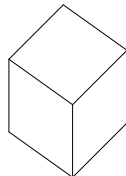
where  $\theta$  is the maximal angle between  $\Gamma_1$  and the adjoining faces.

General facts of such a kind imply various precise regularity statements for special domains. Let us consider for example the flow outside a regular polyhedron  $G$ . On the boundary of  $G$ , the Dirichlet condition is prescribed. Then we obtain  $(u, p) \in W^{2,s} \times W^{1,s}$  on every bounded subdomain of the complement of  $G$ , where the best possible value for  $s$  is (with all digits shown correct)

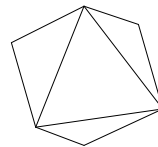
$s = 1.3516\dots$   
for a regular tetrahedron,



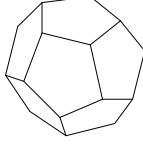
$s = 1.3740\dots$   
for a cube,



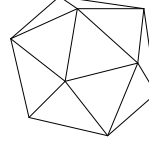
$s = 1.4133\dots$   
for a regular octahedron,



$s = 1.4335\dots$  for a regular dodecahedron,



$s = 1.5248\dots$  for a regular icosahedron



In the last decades, numerous papers appeared which treat boundary value problems for elliptic equations and systems in piecewise smooth domains. For the stationary linear Stokes system see e.g. the references in the book [10]. The properties of solutions of the Dirichlet problem to the nonlinear Navier-Stokes system in 2-dimensional polygonal domains were studied in papers by Kondrat'ev [8], Kellogg and Osborn [7], Kalex [6], Orlt and Sändig [21]. In particular, Kellogg and Osborn proved that the solution of the Dirichlet problem belongs to  $W^{2,2}(\mathcal{G})^3 \times W^{1,2}(\mathcal{G})$  if  $f \in L_2(\mathcal{G})$  and the polygon  $\mathcal{G}$  is convex. Kalex, Orlt and Sändig considered solutions of mixed boundary value problems in polygonal domains. Mixed boundary value problems with boundary conditions (i) and (iii) in 3-dimensional domains with smooth non-intersecting edges were handled by Solonnikov [23], Maz'ya, Plamenevskiĭ and Stupyalis [14]. They proved in particular the solvability in weighted Sobolev and Hölder spaces. For the case of the Dirichlet problem and a polyhedral domain, solvability and regularity results in weighted Sobolev and Hölder spaces were proved by Maz'ya and Plamenevskiĭ [13]. Concerning regularity results in  $L_2$  Sobolev spaces, we refer also to the papers of Nicaise [20] (Dirichlet problem), Ebmeyer and Frehse [4] (mixed problem with boundary conditions (i) and (iii)). Ebmeyer and Frehse proved that  $(u, p) \in W^{s,2}(\mathcal{G})^3 \times W^{s-1,2}(\mathcal{G})$  with arbitrary real  $s < 3/2$  if the angle at the edge, where the boundary conditions change, is less than  $\pi$ . Finally, we mention the papers by Deuring and von Wahl [2], Dindos and Mitrea [3] dealing with the Navier-Stokes system in Lipschitz domains.

The present paper consists of four sections. Section 1 concerns the existence of weak solutions in  $W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ . In Section 2, we introduce and study weighted Sobolev and Hölder space. Here the weights are powers of the distances  $\rho_j$  and  $r_k$  to the vertices and edges of the domain  $\mathcal{G}$ , respectively. In particular, we establish imbedding theorems for these spaces. In contrast to the papers [13, 14, 23], we use weighted spaces with “nonhomogeneous” norms. The weighted Sobolev space  $W_{\beta,\delta}^{l,s}(\mathcal{G})$  in our paper is defined as the set of all functions  $u$  in  $\mathcal{G}$  such that

$$\prod_j \rho_j^{\beta_j - l + |\alpha|} \prod_k \left( \frac{r_k}{\rho} \right)^{\delta_k} \partial_x^\alpha u \in L_s(\mathcal{G})$$

for all  $|\alpha| \leq l$ , where  $\rho = \min_j \rho_j$ . The norm in the weighted Hölder space  $C_{\beta,\delta}^{l,\sigma}(\mathcal{G})$  has a similar structure. The use of weighted spaces with nonhomogeneous norms has several advantages. First, these spaces are applicable to a wider class of boundary value problems. For  $\beta = 0$  and  $\delta = 0$  they are closely related to the nonweighted spaces. Furthermore in some cases (e.g. the Dirichlet problem when the edge angles are less than  $\pi$ ), it is possible to obtain higher regularity results when considering solutions in weighted spaces with nonhomogeneous norms. So we can partially improve the results in [13, 14, 23]. The main results of the paper are contained in Section 3. For the proofs, we use results of our previous papers [18, 19], where we studied mixed boundary value problems for the linear Stokes system. We show in particular that the weak solution  $(u, p)$  belongs to the weighted space  $W_{\beta,\delta}^{2,s}(\mathcal{G})^3 \times W_{\beta,\delta}^{1,s}(\mathcal{G})$  if the data are from corresponding spaces, satisfy certain compatibility conditions, and the numbers  $\frac{5}{2} - \beta_j - \frac{3}{s}$  and  $2 - \delta_k - \frac{2}{s}$  are positive and sufficiently small. The precise conditions on  $\beta$  and  $\delta$  are given in terms of eigenvalues of certain operator pencils. The general results in Section 3 together with estimates for the eigenvalue of these pencils (see [10]) allow us in particular to deduce regularity assertions in nonweighted Sobolev and Hölder spaces. A number of examples is given at the end of Section 3.

The last section concerns the solvability in the space  $W_{\beta,\delta}^{1,s}(\mathcal{G})^3 \times W_{\beta,\delta}^{0,s}(\mathcal{G})$ , where  $s$  may be less than 2. Here, we assume that the Dirichlet condition is given on at least one of the adjoining faces of every edge  $M_k$ . One of our results is the following. Let  $\mathcal{G}$  be an arbitrary polyhedron. Then the problem (0.1) with  $g = 0$  and Dirichlet condition  $u = 0$  on the boundary has a weak solution  $(u, p) \in W^{1,s}(\mathcal{G})^3 \times L_s(\mathcal{G})$  for arbitrary  $f \in W^{-1,s}(\mathcal{G})^3$ ,  $3/2 < s < 3$ , provided the norm of  $f$  is sufficiently small. The same result holds for the mixed problem with boundary condition (i)–(iii) if we suppose that the angles at the edges where the boundary conditions change are less or equal to  $3\pi/2$ .

# 1 Weak solutions of the boundary value problem

## 1.1 The domain

In the following, let  $\mathcal{D}$  be the dihedron

$$\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < r < \infty, -\theta/2 < \varphi < \theta/2, x_3 \in \mathbb{R}\}, \quad (1.1)$$

where  $r, \varphi$  are the polar coordinates in the  $(x_1, x_2)$ -plane,  $r = (x_1^2 + x_2^2)^{1/2}$ ,  $\tan \varphi = x_2/x_1$ . Furthermore, let  $\mathcal{K} = \{x \in \mathbb{R}^3 : x/|x| \in \Omega\}$  be a polyhedral cone with plane faces  $\Gamma_1, \dots, \Gamma_N$  and edges  $M_1, \dots, M_N$ .

The bounded domain  $\mathcal{G} \subset \mathbb{R}^3$  is said to be a *domain of polyhedral type* if

- (i) the boundary  $\partial\mathcal{G}$  consists of smooth (of class  $C^\infty$ ) open two-dimensional manifolds  $\Gamma_j$  (the faces of  $\mathcal{G}$ ),  $j = 1, \dots, N$ , smooth curves  $M_k$  (the edges),  $k = 1, \dots, m$ , and vertices  $x^{(1)}, \dots, x^{(d)}$ ,
- (ii) for every  $\xi \in M_k$  there exist a neighborhood  $\mathcal{U}_\xi$  and a diffeomorphism (a  $C^\infty$  mapping)  $\kappa_\xi$  which maps  $\mathcal{G} \cap \mathcal{U}_\xi$  onto  $\mathcal{D}_\xi \cap B_1$ , where  $\mathcal{D}_\xi$  is a dihedron of the form (1.1) and  $B_1$  is the unit ball,
- (iii) for every vertex  $x^{(j)}$  there exist a neighborhood  $\mathcal{U}_j$  and a diffeomorphism  $\kappa_j$  mapping  $\mathcal{G} \cap \mathcal{U}_j$  onto  $\mathcal{K}_j \cap B_1$ , where  $\mathcal{K}_j$  is a polyhedral cone with vertex at the origin.

The set  $M_1 \cup \dots \cup M_m \cup \{x^{(1)}, \dots, x^{(d)}\}$  of the singular boundary points is denoted by  $\mathcal{S}$ .

## 1.2 Formulation of the problem

For every face  $\Gamma_j$ ,  $j = 1, \dots, N$ , let a number  $d_j \in \{0, 1, 2, 3\}$  be given. We consider the boundary value problem

$$-\nu \Delta u + \sum_{j=1}^3 u_j \partial_{x_j} u + \nabla p = f, \quad -\nabla \cdot u = g \quad \text{in } \mathcal{G}, \quad (1.2)$$

$$S_j u = h_j, \quad N_j(u, p) = \phi_j \quad \text{on } \Gamma_j, \quad j = 1, \dots, N, \quad (1.3)$$

where

$$S_j u = \begin{cases} u & \text{if } d_j = 0, \\ u_\tau & \text{if } d_j = 1, \\ u_n & \text{if } d_j = 2, \end{cases} \quad N_j(u, p) = \begin{cases} -p + 2\nu \varepsilon_{nn} u & \text{if } d_j = 1, \\ \varepsilon_{n\tau}(u) & \text{if } d_j = 2, \\ -pn + 2\nu \varepsilon_n(u) & \text{if } d_j = 3. \end{cases}$$

By a weak solution of the problem (1.2), (1.3), we mean a vector function  $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$  satisfying

$$b(u, v) + \int_{\mathcal{G}} \sum_{j=1}^3 u_j \frac{\partial u}{\partial x_j} \cdot v \, dx - \int_{\mathcal{G}} p \nabla \cdot v \, dx = F(v) \quad \text{for all } v \in V, \quad (1.4)$$

$$-\nabla \cdot u = g \quad \text{in } \mathcal{G}, \quad S_j u = h_j \quad \text{on } \Gamma_j, \quad j = 1, \dots, N, \quad (1.5)$$

where  $V = \{u \in W^{1,2}(\mathcal{G})^3 : S_j u|_{\Gamma_j} = 0, \quad j = 1, \dots, N\}$ ,

$$b(u, v) = 2\nu \int_{\mathcal{G}} \sum_{i,j=1}^3 \varepsilon_{i,j}(u) \varepsilon_{i,j}(v) \, dx, \quad (1.6)$$

$$F(v) = \int_{\mathcal{G}} (f + \nabla g) \cdot v \, dx + \sum_{j=1}^n \int_{\Gamma_j} \phi_j \cdot v \, dx. \quad (1.7)$$

Note that for arbitrary  $u \in W^{1,2}(\mathcal{G})^3$ , the functional  $v \rightarrow \int_{\mathcal{G}} u_j \frac{\partial u}{\partial x_j} \cdot v \, dx$  is continuous on  $W^{1,2}(\mathcal{G})^3$ .

This follows from the inequality

$$\left| \int_{\mathcal{G}} u_j \frac{\partial u}{\partial x_j} \cdot v \, dx \right| \leq \|u_j\|_{L_4(\mathcal{G})} \|\partial_{x_j} u\|_{L_2(\mathcal{G})^3} \|v\|_{L_4(\mathcal{G})^3}$$

and the continuity of the imbedding  $W^{1,2}(\mathcal{G}) \subset L_4(\mathcal{G})$ .

### 1.3 Existence of solutions of the linearized problem

We consider the weak solution of the boundary value problem for the Stokes system

$$-\nu\Delta u + \nabla p = f, \quad \nabla \cdot u = g \quad \text{in } \mathcal{G}, \quad (1.8)$$

i.e. a vector function  $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$  satisfying

$$b(u, v) - \int_{\mathcal{G}} p \nabla \cdot v \, dx = F(v) \quad \text{for all } v \in V, \quad (1.9)$$

$$-\nabla \cdot u = g \quad \text{in } \mathcal{G}, \quad S_j u = h_j \quad \text{on } \Gamma_j, \quad j = 1, \dots, N, \quad (1.10)$$

where  $F$  is given by (1.7). For the proof of the following theorem, we refer to [18, Th.5.1].

**Theorem 1.1** *Let  $g \in L_2(\mathcal{G})$  and  $h_j \in W^{1/2,2}(\Gamma_j)^{3-d_j}$  be such that there exists a vector function  $v \in W^{1,2}(\mathcal{G})^3$ ,  $S_j v = h_j$  on  $\Gamma_j$ ,  $j = 1, \dots, N$ . In the case when  $d_j \in \{0, 2\}$  for all  $j$ , we assume in addition that*

$$\int_{\mathcal{G}} g \, dx + \sum_{j: d_j=0} \int_{\Gamma_j} h_j \cdot n \, dx + \sum_{j: d_j=2} \int_{\Gamma_j} h_j \, dx = 0. \quad (1.11)$$

Furthermore, let the functional  $F \in V^*$  satisfy the condition

$$F(v) = 0 \quad \text{for all } v \in L_V, \quad (1.12)$$

where  $L_V$  denotes the set of all  $v \in V$  such that  $\varepsilon_{i,j}(v) = 0$  for  $i, j = 1, 2, 3$ . Then there exists a solution  $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$  of the problem (1.9), (1.10). Here  $p$  is uniquely determined if  $d_j \in \{1, 3\}$  for at least one  $j$  and unique up to constants if  $d_j \in \{0, 2\}$  for all  $j$ . The vector function  $u$  is unique up to elements from  $L_V$ .

Note that  $L_V$  contains only functions of the form  $v = c + Ax$ , where  $c$  is a constant vector and  $A$  is a constant matrix,  $A = -A^t$  (rigid body motions). In particular,  $\nabla \cdot v = 0$  for  $v \in L_V$ . In most cases (e.g. if the Dirichlet condition is given on at least one face  $\Gamma_j$ ), the set  $L_V$  contains only the function  $v = 0$ .

### 1.4 Existence of solutions of the nonlinear problem

Let the operator  $Q$  be defined by

$$Qu = (u \cdot \nabla) u.$$

Obviously,  $Q$  realizes a mapping  $W^{1,2}(\mathcal{G}) \rightarrow V^*$ . Furthermore, there exist constants  $c_1, c_2$  such that

$$\|Qu\|_{V^*} \leq c \|u\|_{W^{1,2}(\mathcal{G})}^2 \quad \text{for all } u \in W^{1,2}(\mathcal{G}), \quad (1.13)$$

$$\|Qu - Qv\|_{V^*} \leq c (\|u\|_{W^{1,2}(\mathcal{G})} + \|v\|_{W^{1,2}(\mathcal{G})}) \|u - v\|_{W^{1,2}(\mathcal{G})} \quad \text{for all } u, v \in W^{1,2}(\mathcal{G}). \quad (1.14)$$

Using the last two estimates together with Theorem 1.1, we can prove the following statement.

**Theorem 1.2** *Let  $g$  and  $h_j$  be as in Theorem 1.1. Furthermore, we suppose that  $L_V = \{0\}$  and*

$$\|F\|_{V^*} + \|g\|_{L_2(\mathcal{G})} + \sum_{j=1}^N \|h_j\|_{W^{1/2,2}(\Gamma_j)^{3-d_j}}$$

*is sufficiently small. Then there exists a solution  $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$  of the problem (1.4), (1.5). Here,  $u$  is unique on the set of all functions with norm less than a certain positive  $\varepsilon$ ,  $p$  is unique if  $d_j \in \{1, 3\}$  for at least one  $j$ , otherwise  $p$  is unique up to a constant.*

**P r o o f.** Let  $(u^{(0)}, p^{(0)}) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$  be the solution of the linear problem (1.9), (1.10). By our assumptions on  $F$ ,  $g$  and  $h_j$ , we may assume that

$$\|u^{(0)}\|_{W^{1,2}(\mathcal{G})^3} \leq \varepsilon_1,$$

where  $\varepsilon_1$  is a small positive number. Let  $V_0$  denote the set of all  $v \in V$  such that  $\nabla \cdot v = 0$ . We put  $w = u - u^{(0)}$  and  $q = p - p^{(0)}$ . Then  $(u, p)$  is a solution of the problem (1.4), (1.5) if and only if  $(w, q) \in V_0 \times L_2(\mathcal{G})$  and

$$b(w, v) - \int_{\mathcal{G}} q \nabla \cdot v \, dx = - \int_{\mathcal{G}} Q(w + u^{(0)}) \cdot v \, dx \quad \text{for all } v \in V. \quad (1.15)$$

By Theorem 1.1, there is a linear and continuous mapping

$$V^* \ni \Phi \rightarrow A\Phi = (w, q) \in V_0 \times L_2(\mathcal{G})$$

defined by

$$b(w, v) - \int_{\mathcal{G}} q \nabla \cdot v \, dx = \Phi(v) \quad \text{for all } v \in V, \quad \int_{\mathcal{G}} q \, dx = 0 \text{ if } d_j \in \{0, 2\} \text{ for all } j.$$

We write (1.15) as

$$(w, q) = T(w, q), \quad \text{where } T(w, q) = -AQ(w + u^{(0)}).$$

Due to (1.14), the operator  $T$  is contractive on the set of all  $(w, q) \in V_0 \times L_2(\mathcal{G})$  with norm  $\leq \varepsilon_2$  if  $\varepsilon_1$  and  $\varepsilon_2$  are sufficiently small. Hence there exist  $w \in W^{1,2}(\mathcal{G})^3$  and  $q \in L_2(\mathcal{G})$  satisfying (1.15). The result follows.  $\square$

**Remark 1.1** If  $d_j \in \{0, 2\}$  for all  $j$ , then

$$\int_{\mathcal{G}} \sum_{j=1}^3 u_j \frac{\partial v}{\partial x_j} \cdot v \, dx = 0 \quad \text{for all } v \in W^{1,2}(\mathcal{G})^3, \quad u \in V, \quad \nabla \cdot u = 0 \quad (1.16)$$

(see [5, Le.IV.2.2]). Thus, analogously to [5, Th.IV.2.3] (see also [24]), the problem (1.4), (1.5) has at least one solution for arbitrary  $F \in V^*$ ,  $g = 0$ ,  $h_j \in W^{1/2,2}(\Gamma_j)^{3-d_j}$  satisfying (1.11).

## 2 Weighted Sobolev and Hölder spaces

Here, we introduce weighted Sobolev and Hölder spaces in polyhedral domains and prove imbeddings for these spaces which will be used in the next section. We start with the case of a polyhedral cone.

### 2.1 Weighted Sobolev spaces in a cone

Let  $\mathcal{K} = \{x \in \mathbb{R}^3 : x/|x| \in \Omega\}$  be a polyhedral cone in  $\mathbb{R}^3$  whose boundary consists of plane faces  $\Gamma_j$  and edges  $M_k$ ,  $j, k = 1, \dots, N$ . We denote by  $\rho(x) = |x|$  the distance of  $x$  to the vertex of the cone, by  $r_k(x)$  the distance to the edge  $M_k$ , and by  $r(x)$  the distance to the set  $\mathcal{S} = M_1 \cup \dots \cup M_N \cup \{0\}$ . Note that there exist positive constants  $c_1, c_2$  such that

$$c_1 r(x) \leq \rho(x) \prod_{k=1}^N \frac{r_k(x)}{\rho(x)} \leq c_2 r(x) \quad \text{for all } x \in \mathcal{K}. \quad (2.1)$$

Let  $l$  be a nonnegative integer,  $\beta \in \mathbb{R}$ ,  $\delta = (\delta_1, \dots, \delta_N) \in \mathbb{R}^N$ , and  $1 < s < \infty$ . We define  $V_{\beta, \delta}^{l, s}(\mathcal{K})$  as the closure of the set  $C_0^\infty(\overline{\mathcal{K}} \setminus \mathcal{S})$  with respect to the norm

$$\|u\|_{V_{\beta, \delta}^{l, s}(\mathcal{K})} = \left( \int_{\mathcal{K}} \sum_{|\alpha| \leq l} \rho^{s(\beta - l + |\alpha|)} \prod_{k=1}^N \left( \frac{r_k}{\rho} \right)^{s(\delta_k - l + |\alpha|)} |\partial_x^\alpha u|^s \, dx \right)^{1/s}.$$

The weighted Sobolev space  $W_{\beta,\delta}^{l,s}(\mathcal{K})$ , where  $\delta_k > -2/s$  for  $k = 1, \dots, N$ , is defined as the closure of the set  $C_0^\infty(\overline{\mathcal{K}})$  with respect to the norm

$$\|u\|_{W_{\beta,\delta}^{l,s}(\mathcal{K})} = \left( \int_{\mathcal{K}} \sum_{|\alpha| \leq l} \rho^{s(\beta-l+|\alpha|)} \prod_{k=1}^N \left( \frac{r_k}{\rho} \right)^{s\delta_k} dx \right)^{1/s}.$$

If  $\delta$  is a real number, then by  $V_{\beta,\delta}^{l,s}(\mathcal{K})$  and  $W_{\beta,\delta}^{l,s}(\mathcal{K})$ , we mean the above introduced spaces with  $\delta_1 = \dots = \delta_N = \delta$ . For the proof of the following lemma we refer to [16, Le.1].

**Lemma 2.1** *Let  $1 < s \leq t < \infty$ ,  $l-3/s \geq l'-3/t$ ,  $\beta-l+3/s = \beta'-l'+3/t$ , and  $\delta_k-l+3/s \leq \delta'_k-l'+3/t$  for  $k = 1, \dots, N$ . Then  $V_{\beta,\delta}^{l,s}(\mathcal{K})$  is continuously imbedded into  $V_{\beta',\delta'}^{l',t}(\mathcal{K})$ .*

In particular, we have  $V_{\beta,\delta}^{l,s}(\mathcal{K}) \subset V_{\beta',\delta'}^{l',s}(\mathcal{K})$  for  $l \geq l'$ ,  $\beta-l = \beta'-l'$  and  $\delta_k-l \leq \delta'_k-l'$ ,  $k = 1, \dots, N$ . If in addition  $\delta_k > -2/s$  and  $\delta'_k > -2/s$  for  $k = 1, \dots, N$ , then also

$$W_{\beta,\delta}^{l,s}(\mathcal{K}) \subset W_{\beta',\delta'}^{l',s}(\mathcal{K}).$$

The spaces  $V_{\beta,\delta}^{l,s}(\mathcal{K})$  and  $W_{\beta,\delta}^{l,s}(\mathcal{K})$  coincide if  $\delta_k > l-2/s$  for all  $k$  (see [18]).

**Lemma 2.2** *Let  $1 < s \leq t < \infty$  and  $l-3/s \geq \max(\delta_k, 0) - 3/t$ . Then  $W_{\beta,\delta}^{l,s}(\mathcal{K}) \subset W_{\beta-l+3/s-3/t,0}^{0,t}(\mathcal{K})$  and*

$$\|\rho^{\beta-l+3/s-3/t}u\|_{L_t(\mathcal{K})} \leq c \|u\|_{W_{\beta,\delta}^{l,s}(\mathcal{K})} \quad (2.2)$$

for all  $u \in W_{\beta,\delta}^{l,s}(\mathcal{K})$  with a constant  $c$  independent of  $u$ . Furthermore, for arbitrary  $u \in W_{\beta,\delta}^{l,s}(\mathcal{K})$ ,  $l-3/s > \max(\delta_k, 0)$ , the following inequality is valid.

$$\|\rho^{\beta-l+3/s}u\|_{L_\infty(\mathcal{K})} \leq c \|u\|_{W_{\beta,\delta}^{l,s}(\mathcal{K})} \quad (2.3)$$

**P r o o f.** Let  $u \in W_{\beta,\delta}^{l,s}(\mathcal{K})$ , and let  $\zeta_k$  be infinitely differentiable functions with support in  $\{x : 2^{k-1} < |x| < 2^{k+1}\}$  such that

$$|\partial_x^\alpha \zeta_k(x)| < c 2^{-k|\alpha|} \quad \text{for } k \leq l, \quad \sum_{k=-\infty}^{+\infty} \zeta_k = 1.$$

We define  $v(x) = u(2^k x)$  and  $\eta_k(x) = \zeta_k(2^k x)$ . Then  $\eta_k(x)$  vanishes for  $|x| < 1/2$  and  $|x| > 2$ . Therefore for  $l' = l - \max(\delta_k, 0)$ , we have

$$\|\eta_k v\|_{W^{l',s}(\mathcal{K})} \leq c \|\eta_k v\|_{W_{\beta,\delta}^{l,s}(\mathcal{K})}$$

(see [22, Th.3]). This inequality together with the continuity of the imbedding  $W^{l',s} \subset L_t$  implies

$$\|\eta_k v\|_{L_t(\mathcal{K})} \leq c \|\eta_k v\|_{W_{\beta,\delta}^{l,s}(\mathcal{K})},$$

where  $c$  is independent of  $u$  and  $k$ . Using the equalities

$$\|\eta_k v\|_{L_t(\mathcal{K})} = 2^{-3k/t} \|\zeta_k u\|_{L_t(\mathcal{K})} \quad \text{and} \quad \|\eta_k v\|_{W_{\beta,\delta}^{l,s}(\mathcal{K})} = 2^{-k(\beta-l)-3k/s} \|\zeta_k u\|_{W_{\beta,\delta}^{l,s}(\mathcal{K})},$$

we obtain

$$\|\rho^{\beta-l+3/s-3/t} \zeta_k u\|_{L_t(\mathcal{K})} \leq c \|\zeta_k u\|_{W_{\beta,\delta}^{l,s}(\mathcal{K})}.$$

Consequently,

$$\begin{aligned} \|\rho^{\beta-l+3/s-3/t}u\|_{L_t(\mathcal{K})} &\leq c \left( \sum_{k=-\infty}^{+\infty} \|\rho^{\beta-l+3/s-3/t} \zeta_k u\|_{L_t(\mathcal{K})}^t \right)^{1/t} \leq c \left( \sum_{k=-\infty}^{+\infty} \|\zeta_k u\|_{W_{\beta,\delta}^{l,s}(\mathcal{K})}^t \right)^{1/t} \\ &\leq c \left( \sum_{k=-\infty}^{+\infty} \|\zeta_k u\|_{W_{\beta,\delta}^{l,s}(\mathcal{K})}^s \right)^{1/s} \leq c \|\zeta_k u\|_{W_{\beta,\delta}^{l,s}(\mathcal{K})}. \end{aligned}$$

This proves (2.2). The proof of (2.3) proceeds analogously.  $\square$

**Lemma 2.3** *Let  $u \in W_{\beta,\delta}^{l,s}(\mathcal{K})$ , and let  $j$  be an integer,  $j \geq 1$ . Then there exist functions  $v \in V_{\beta,\delta}^{l,s}(\mathcal{K})$  and  $w \in W_{\beta+j,\delta+j}^{l+j,s}(\mathcal{K})$  such that  $u = v + w$  and*

$$\|v\|_{V_{\beta,\delta}^{l,s}(\mathcal{K})} + \|w\|_{W_{\beta+j,\delta+j}^{l+j,s}(\mathcal{K})} \leq c \|u\|_{W_{\beta,\delta}^{l,s}(\mathcal{K})}.$$

*P r o o f.* Let  $\zeta_k$  be the same functions as in the proof of Lemma 2.2, and let  $u$  be an arbitrary function from  $W_{\beta,\delta}^{l,s}(\mathcal{K})$ . Obviously, the function  $\tilde{u}_k$  defined by  $\tilde{u}_k(x) = \zeta_k(2^k x) u(2^k x)$  belongs also to  $W_{\beta,\delta}^{l,s}(\mathcal{K})$  and vanishes for  $|x| < 1/2$  and  $|x| > 2$ . Consequently by [15, Le.1.3] (for integer  $\beta + 2/s$  see [22, Th.5]), there exist functions  $\tilde{v}_k \in V_{\beta,\delta}^{l,s}(\mathcal{K})$  and  $\tilde{w}_k \in W_{\beta+j,\delta+j}^{l+j,s}(\mathcal{K})$  with supports in  $\{x : 1/4 < |x| < 4\}$  such that  $\tilde{u}_k = \tilde{v}_k + \tilde{w}_k$  and

$$\|\tilde{v}_k\|_{V_{\beta,\delta}^{l,s}(\mathcal{K})} + \|\tilde{w}_k\|_{W_{\beta+j,\delta+j}^{l+j,s}(\mathcal{K})} \leq c \|\tilde{u}_k\|_{W_{\beta,\delta}^{l,s}(\mathcal{K})},$$

where  $c$  is independent of  $u$  and  $k$ . Let  $v_k(x) = \tilde{v}_k(2^{-k}x)$  and  $w_k(x) = \tilde{w}_k(2^{-k}x)$ . Then the supports of  $v_k$  and  $w_k$  are contained in  $\{x : 2^{k-2} < |x| < 2^{k+2}\}$ , and we have  $\zeta_k u = v_k + w_k$  for all  $k$ . Moreover,

$$\begin{aligned} \|v_k\|_{V_{\beta,\delta}^{l,s}(\mathcal{K})} + \|w_k\|_{W_{\beta+j,\delta+j}^{l+j,s}(\mathcal{K})} &= 2^{k(\beta-l+3/s)} \left( \|\tilde{v}_k\|_{V_{\beta,\delta}^{l,s}(\mathcal{K})} + \|\tilde{w}_k\|_{W_{\beta+j,\delta+j}^{l+j,s}(\mathcal{K})} \right) \\ &\leq c 2^{k(\beta-l+3/s)} \|\tilde{u}_k\|_{W_{\beta,\delta}^{l,s}(\mathcal{K})} = c \|\zeta_k u\|_{W_{\beta,\delta}^{l,s}(\mathcal{K})}. \end{aligned}$$

Let  $v = \sum_{k=-\infty}^{+\infty} v_k$  and  $w = \sum_{k=-\infty}^{+\infty} w_k$ . Then  $u = v + w$ . Since  $v_k$  and  $v_m$  have disjoint supports for  $|k - m| \geq 4$ , we have  $v \in V_{\beta,\delta}^{l,s}(\mathcal{K})$  and

$$\|v\|_{V_{\beta,\delta}^{l,s}(\mathcal{K})}^s \leq 7^{s-1} \sum_k \|v_k\|_{V_{\beta,\delta}^{l,s}(\mathcal{K})}^s \leq c \sum_k \|\zeta_k u\|_{W_{\beta,\delta}^{l,s}(\mathcal{K})}^s \leq c' \|u\|_{W_{\beta,\delta}^{l,s}(\mathcal{K})}^s.$$

Analogously, the norm of  $w$  in  $W_{\beta+j,\delta+j}^{l+j,s}(\mathcal{K})$  can be estimated by the norm of  $u$  in  $W_{\beta,\delta}^{l,s}(\mathcal{K})$ . The lemma is proved.  $\square$

## 2.2 Weighted Hölder spaces in a polyhedral cone

Let  $l$  be a nonnegative integer,  $\beta \in \mathbb{R}$ ,  $\delta = (\delta_1, \dots, \delta_N) \in \mathbb{R}^N$ , and  $\sigma \in (0, 1)$ . We define the weighted Hölder space  $\mathcal{N}_{\beta,\delta}^{l,\sigma}(\mathcal{K})$  as the set of all  $l$  times continuously differentiable functions on  $\bar{\mathcal{K}} \setminus \mathcal{S}$  with finite norm

$$\begin{aligned} \|u\|_{\mathcal{N}_{\beta,\delta}^{l,\sigma}(\mathcal{K})} &= \sum_{|\alpha| \leq l} \sup_{x \in \mathcal{K}} |x|^{\beta-l-\sigma+|\alpha|} \prod_{k=1}^N \left( \frac{r_k(x)}{|x|} \right)^{\delta_k-l-\sigma+|\alpha|} |\partial_x^\alpha u(x)| \\ &\quad + \sum_{|\alpha|=l} \sup_{|x-y| < r(x)/2} |x|^\beta \prod_{k=1}^N \left( \frac{r_k(x)}{|x|} \right)^{\delta_k} \frac{|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)|}{|x-y|^\sigma}. \end{aligned} \quad (2.4)$$

Furthermore, the space  $C_{\beta,\delta}^{l,\sigma}(\mathcal{K})$  is defined for nonnegative  $\delta_k$ ,  $k = 1, \dots, N$ , as the set of all  $l$  times continuously differentiable functions on  $\bar{\mathcal{K}} \setminus \mathcal{S}$  with finite norm

$$\begin{aligned} \|u\|_{C_{\beta,\delta}^{l,\sigma}(\mathcal{K})} &= \sum_{|\alpha| \leq l} \sup_{x \in \mathcal{K}} |x|^{\beta-l-\sigma+|\alpha|} \prod_{k=1}^N \left( \frac{r_k(x)}{|x|} \right)^{\max(0, \delta_k-l-\sigma+|\alpha|)} |\partial_x^\alpha u(x)| \\ &\quad + \sum_{k: \sigma_k \leq l} \sum_{|\alpha|=l-\sigma_k} \sup_{\substack{x,y \in \mathcal{K}_k \\ |x-y| < |x|/2}} |x|^{\beta-\delta_k} \frac{|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)|}{|x-y|^{\sigma+\sigma_k-\delta_k}} \\ &\quad + \sum_{|\alpha|=l} \sup_{\substack{x,y \in \mathcal{K} \\ |x-y| < r(x)/2}} |x|^\beta \prod_{k=1}^N \left( \frac{r_k(x)}{|x|} \right)^{\delta_k} \frac{|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)|}{|x-y|^\sigma}, \end{aligned} \quad (2.5)$$



where  $\mathcal{K}_k = \{x \in \mathcal{K} : r_k(x) < 3r(x)/2\}$ ,  $\sigma_k = [\delta_k - \sigma] + 1$ ,  $[s]$  denotes the greatest integer less or equal to  $s$ . The trace spaces for  $\mathcal{N}_{\beta,\delta}^{l,\sigma}(\mathcal{K})$  and  $C_{\beta,\delta}^{l,\sigma}(\mathcal{K})$  on  $\Gamma_j$  are denoted by  $\mathcal{N}_{\beta,\delta}^{l,\sigma}(\Gamma_j)$  and  $C_{\beta,\delta}^{l,\sigma}(\Gamma_j)$ , respectively.

Obviously,  $\mathcal{N}_{\beta,\delta}^{l,\sigma}(\mathcal{K})$  is a subset of  $C_{\beta,\delta}^{l,\sigma}(\mathcal{K})$ . If  $\delta_k \geq l + \sigma$  for  $k = 1, \dots, N$ , then both spaces coincide. Furthermore, the following imbedding holds (and is continuous, see [19]).

$$\mathcal{N}_{\beta,\delta}^{l,\sigma}(\mathcal{K}) \subset \mathcal{N}_{\beta',\delta'}^{l',\sigma'}(\mathcal{K}) \quad \text{if } l + \sigma \geq l' + \sigma', \quad \beta - l - \sigma = \beta' - l' - \sigma', \quad \delta_k - l - \sigma \leq \delta'_k - l' - \sigma'.$$

If in addition  $\delta_k$  and  $\delta'_k$  are nonnegative, then  $C_{\beta,\delta}^{l,\sigma}(\mathcal{K})$  is continuously imbedded into  $C_{\beta',\delta'}^{l',\sigma'}(\mathcal{K})$ . Next, we prove a relation between the spaces  $V_{\beta,\delta}^{l,\sigma}$  and  $\mathcal{N}_{\beta,\delta}^{l,\sigma}$ .

**Lemma 2.4** *Suppose that  $l - 3/s > l' + \sigma$ ,  $\beta - l + 3/s = \beta' - l' - \sigma$  and  $\delta_k - l + 3/s \leq \delta'_k - l' - \sigma$  for  $k = 1, \dots, N$ . Then  $V_{\beta,\delta}^{l,\sigma}(\mathcal{K})$  is continuously imbedded into  $\mathcal{N}_{\beta',\delta'}^{l',\sigma'}(\mathcal{K})$ .*

**P r o o f.** It suffices to prove the lemma for  $\delta_k - l + 3/s = \delta'_k - l' - \sigma$ ,  $k = 1, \dots, N$ . Let  $u \in V_{\beta,\delta}^{l,\sigma}(\mathcal{K})$ . For an arbitrary point  $x \in \mathcal{K}$ , we denote by  $B_x$  the set  $\{x' \in \mathcal{K} : |x - x'| < r(x)/2\}$ . Note that

$$|x|/2 \leq |x'| < 3|x|/2, \quad r_k(x)/2 \leq r_k(x') < 3r_k(x)/2, \quad r(x)/2 \leq r(x') \leq 3r(x)/2 \quad \text{for } x' \in B_x. \quad (2.6)$$

First, let  $r(x) = 1$ . From the continuity of the imbedding  $W^{l,s}(B_x) \subset C^{l',\sigma}(B_x)$  it follows that there exists a constant  $c$  independent of  $u$  and  $x$  such that

$$\begin{aligned} |(\partial^\alpha u)(x)| &\leq c \|u\|_{W^{l,s}(B_x)} \quad \text{for } |\alpha| \leq l', \\ \frac{|(\partial^\alpha u)(x) - (\partial^\alpha u)(x')|}{|x - x'|^\sigma} &\leq c \|u\|_{W^{l,s}(B_x)} \quad \text{for } |\alpha| = l', \quad x' \in B_x. \end{aligned}$$

Due to (2.1) and (2.6), this implies

$$|x|^\beta \prod_k \left( \frac{r_k(x)}{|x|} \right)^{\delta_k} |(\partial^\alpha u)(x)| \leq c \sum_{|\gamma| \leq l} \left\| r^{|\gamma|-l} \rho^\beta \prod_k \left( \frac{r_k}{\rho} \right)^{\delta_k} \partial^\gamma u \right\|_{L^s(B_x)} \leq c \|u\|_{V_{\beta,\delta}^{l,s}(\mathcal{K})} \quad \text{for } |\alpha| \leq l'$$

and analogously

$$|x|^\beta \prod_k \left( \frac{r_k(x)}{|x|} \right)^{\delta_k} \frac{|(\partial^\alpha u)(x) - (\partial^\alpha u)(x')|}{|x - x'|^\sigma} \leq c \|u\|_{V_{\beta,\delta}^{l,s}(\mathcal{K})} \quad \text{for } |\alpha| = l', \quad x' \in B_x.$$

Now let  $x$  be an arbitrary point in  $\mathcal{K}$  and  $x' \in B_x$ . We put  $y = x/r(x)$ ,  $y' = x'/r(x)$ . Then  $r(y) = 1$  and  $y' \in B_y$ . Consequently, the function  $v(\xi) = u(r(x)\xi)$  satisfies the inequalities

$$|y|^\beta \prod_k \left( \frac{r_k(y)}{|y|} \right)^{\delta_k} |(\partial^\alpha v)(y)| \leq c \|v\|_{V_{\beta,\delta}^{l,s}(\mathcal{K})} \leq c r(x)^{l-\beta-3/s} \|u\|_{V_{\beta,\delta}^{l,s}(\mathcal{K})}$$

for  $|\alpha| \leq l'$  and

$$|y|^\beta \prod_k \left( \frac{r_k(y)}{|y|} \right)^{\delta_k} \frac{|(\partial^\alpha v)(y) - (\partial^\alpha v)(y')|}{|y - y'|^\sigma} \leq c r(x)^{l-\beta-3/s} \|u\|_{V_{\beta,\delta}^{l,s}(\mathcal{K})}.$$

for  $|\alpha| = l'$ . Here, the constant  $c$  is independent of  $u$ ,  $x$  and  $x'$ . Using (2.1), we obtain the inequalities

$$\begin{aligned} |x|^{\beta-l+|\alpha|+3/s} \prod_k \left( \frac{r_k(x)}{|x|} \right)^{\delta_k-l+|\alpha|+3/s} |(\partial^\alpha u)(x)| &\leq c \|u\|_{V_{\beta,\delta}^{l,s}(\mathcal{K})} \quad \text{for } |\alpha| \leq l', \\ |x|^{\beta'} \prod_k \left( \frac{r_k(x)}{|x|} \right)^{\delta_k-l+l'+\sigma+3/s} \frac{|(\partial^\alpha u)(x) - (\partial^\alpha u)(x')|}{|x - x'|^\sigma} &\leq c \|u\|_{V_{\beta,\delta}^{l,s}(\mathcal{K})} \quad \text{for } |\alpha| = l'. \end{aligned}$$

The result follows.  $\square$

**Corollary 2.1** *Let  $u \in W_{\beta,\delta}^{l,s}(\mathcal{K})$ ,  $l > 3/s$ . Then*

$$\rho^{\beta-l+3/s} \prod_k \left(\frac{r_k}{\rho}\right)^{\sigma_k} u \in L_\infty(\mathcal{K}), \quad (2.7)$$

where  $\sigma_k = 0$  for  $\delta_k < l - 3/s$ ,  $\sigma_k = 1/s + \varepsilon$  for  $l - 3/s \leq \delta_k \leq l - 2/s$ , and  $\sigma_k = \delta_k - l + 3/s$  for  $\delta_k > l - 2/s$  ( $\varepsilon$  is an arbitrarily small positive number).

**P r o o f.** Let  $\psi_k$  be a smooth function on the unit sphere  $S^2$  such that  $\psi_k = \delta_{j,k}$  in a neighborhood of the points  $S^2 \cap M_j$  for  $j = 1, \dots, m$ . We extend  $\psi_k$  to  $\mathbb{R}^3 \setminus \{0\}$  by  $\psi_k(x) = \psi_k(x/|x|)$ . Then  $\psi_k u \in W_{\beta,\delta_k}^{l,s}(\mathcal{K})$ . Obviously, it suffices to prove (2.7) for the function  $\psi_k u$ . If  $\delta_k < l - 3/s$ , then by Lemma 2.2,  $\rho^{\beta-l+3/s} \psi_k u \in L_\infty(\mathcal{K})$ . If  $\delta_k > l - 2/s$ , then  $W_{\beta,\delta_k}^{l,s}(\mathcal{K}) = V_{\beta,\delta_k}^{l,s}(\mathcal{K})$ . By Lemma 2.4, the last space is imbedded into  $\mathcal{N}_{\beta-l+\sigma+3/s, \delta_k-l+\sigma+3/s}^{0,\sigma}(\mathcal{K})$  for arbitrary  $\sigma < l - 3/s$ . Therefore in particular,

$$\rho^{\beta-l+3/s} \prod_k \left(\frac{r_k}{\rho}\right)^{\delta_k-l+3/s} \psi_k u \in L_\infty(\mathcal{K}).$$

For  $l - 3/s \leq \delta_k < l - 2/s$ , the assertion follows from the imbedding  $W_{\beta,\delta_k}^{l,s}(\mathcal{K}) \subset W_{\beta, l-2/s+\varepsilon}^{l,s}(\mathcal{K})$ .  $\square$

### 2.3 Weighted Sobolev and Hölder spaces in a bounded polyhedral domain

Let  $\mathcal{G}$  be a domain of polyhedral type (see Section 1.1) with faces  $\Gamma_1, \dots, \Gamma_N$ , edges  $M_1, \dots, M_m$  and vertices  $x^{(1)}, \dots, x^{(d)}$ . We denote the distance of  $x$  to the edge  $M_k$  by  $r_k(x)$ , the distance to the vertex  $x^{(j)}$  by  $\rho_j(x)$ , the distance to  $\mathcal{S}$  (the set of all edge points and vertices) by  $r(x)$ , and the distance to the set  $X = \{x^{(1)}, \dots, x^{(d)}\}$  by  $\rho(x)$ . For arbitrary integer  $l \geq 0$ , real  $s > 1$  and real tuples  $\beta = (\beta_1, \dots, \beta_d)$ ,  $\delta = (\delta_1, \dots, \delta_m)$ , we define  $V_{\beta,\delta}^{l,s}(\mathcal{G})$  and  $W_{\beta,\delta}^{l,s}(\mathcal{G})$  as the weighted Sobolev spaces with the norms

$$\begin{aligned} \|u\|_{V_{\beta,\delta}^{l,s}(\mathcal{G})} &= \left( \int_{\mathcal{K}} \sum_{|\alpha| \leq l} \prod_{j=1}^d \rho_j^{s(\beta_j-l+|\alpha|)} \prod_{k=1}^m \left(\frac{r_k}{\rho}\right)^{s(\delta_k-l+|\alpha|)} |\partial_x^\alpha u|^s dx \right)^{1/s}, \\ \|u\|_{W_{\beta,\delta}^{l,s}(\mathcal{K})} &= \left( \int_{\mathcal{K}} \sum_{|\alpha| \leq l} \prod_{j=1}^d \rho_j^{s(\beta_j-l+|\alpha|)} \prod_{k=1}^m \left(\frac{r_k}{\rho}\right)^{s\delta_k} |\partial_x^\alpha u|^s dx \right)^{1/s}, \end{aligned}$$

respectively. In the case of the space  $W_{\beta,\delta}^{l,s}(\mathcal{G})$ , we suppose that  $\delta_k > -2/s$  for  $k = 1, \dots, m$ . The corresponding trace spaces on the faces  $\Gamma_j$  are denoted by  $V_{\beta,\delta}^{l-1/s,s}(\Gamma_j)$  and  $W_{\beta,\delta}^{l-1/s,s}(\Gamma_j)$ , respectively. Furthermore, let  $V_{\beta,\delta}^{-1,s}(\mathcal{G})$  denote the dual space of  $V_{-\beta,-\delta}^{1,s'}(\mathcal{G})$ ,  $s' = s/(s-1)$ , with respect to the  $L_2$  scalar product.

**Lemma 2.5** *Let  $1 < t < s < \infty$ ,  $\beta_j + 3/s < \beta'_j + 3/t$  for  $j = 1, \dots, d$ , and  $\delta_k + 2/s < \delta'_k + 2/t$  for  $k = 1, \dots, N$ . Then  $V_{\beta,\delta}^{l,s}(\mathcal{G}) \subset V_{\beta',\delta'}^{l,t}(\mathcal{G})$  and  $W_{\beta,\delta}^{l,s}(\mathcal{G}) \subset W_{\beta',\delta'}^{l,t}(\mathcal{G})$ . These imbeddings are continuous.*

**P r o o f.** Let  $q = st/(s-t)$ . By Hölder's inequality,

$$\left\| \prod_j \rho_j^{\beta'_j-l+|\alpha|} \prod_k \left(\frac{r_k}{\rho}\right)^{\delta'_k} \partial_x^\alpha u \right\|_{L_t(\mathcal{G})} \leq c \left\| \prod_j \rho_j^{\beta_j-l+|\alpha|} \prod_k \left(\frac{r_k}{\rho}\right)^{\delta_k} \partial_x^\alpha u \right\|_{L_t(\mathcal{G})},$$

where

$$c = \left\| \prod_j \rho_j^{\beta'_j-\beta_j} \prod_k \left(\frac{r_k}{\rho}\right)^{\delta'_k-\delta_k} \right\|_{L_q(\mathcal{G})} < \infty$$

if  $\beta'_j - \beta_j > -3/q$  and  $\delta'_k - \delta_k > -2/q$ . This proves the imbedding  $W_{\beta,\delta}^{l,s}(\mathcal{G}) \subset W_{\beta',\delta'}^{l,t}(\mathcal{G})$ . Analogously, the imbedding  $V_{\beta,\delta}^{l,s}(\mathcal{G}) \subset V_{\beta',\delta'}^{l,t}(\mathcal{G})$  holds.  $\square$

The following result can be directly deduced from Lemma 2.1.

**Lemma 2.6** Let  $1 < s \leq t < \infty$ ,  $l - 3/s \geq l' - 3/t$ ,  $\beta_j - l + 3/s \leq \beta'_j - l' + 3/t$  for  $j = 1, \dots, d$ , and  $\delta_k - l + 3/s \leq \delta'_k - l' + 3/t$  for  $k = 1, \dots, m$ . Then  $V_{\beta, \delta}^{l, s}(\mathcal{G})$  is continuously imbedded into  $V_{\beta', \delta'}^{l', t}(\mathcal{G})$ .

**Corollary 2.2** Let  $1 < s \leq t < \infty$ ,  $3/s \leq 1 + 3/t$ ,  $\beta_j + 3/s \leq \beta'_j + 1 + 3/t$  for  $j = 1, \dots, d$ , and  $\delta_k + 3/s \leq \delta'_k + 1 + 3/t$  for  $k = 1, \dots, m$ . Then  $V_{\beta, \delta}^{0, s}(\mathcal{G})$  is continuously imbedded into  $V_{\beta', \delta'}^{-1, t}(\mathcal{G})$ .

**P r o o f.** According to Lemma 2.6, we have  $V_{-\beta', -\delta'}^{1, t'}(\mathcal{G}) \subset V_{-\beta, -\delta}^{0, s'}(\mathcal{G})$ , where  $s' = s/(s-1)$ ,  $t' = t/(t-1)$ . The result follows.  $\square$

Let us further note that (as in the case of a cone)

$$W_{\beta, \delta}^{l, s}(\mathcal{G}) \subset W_{\beta', \delta'}^{l', s}(\mathcal{G}) \quad \text{if } l \geq l', \beta_j - l \leq \beta'_j - l', \delta_k - l \leq \delta'_k - l', \delta_k > -2/s, \delta'_k > -2/s$$

for  $j = 1, \dots, d, k = 1, \dots, m$ . If  $\delta_k > l - 2/s$  for  $k = 1, \dots, m$ , then  $V_{\beta, \delta}^{l, s}(\mathcal{G}) = W_{\beta, \delta}^{l, s}(\mathcal{G})$ .

We introduce the following weighted Hölder spaces in the domain  $\mathcal{G}$ . The space  $\mathcal{N}_{\beta, \delta}^{l, \sigma}(\mathcal{G})$  is defined as the set all  $l$  times continuously differentiable functions on  $\bar{\mathcal{G}} \setminus \mathcal{S}$  with finite norm

$$\begin{aligned} \|u\|_{\mathcal{N}_{\beta, \delta}^{l, \sigma}(\mathcal{G})} &= \sum_{|\alpha| \leq l} \sup_{x \in \mathcal{G}} \prod_{j=1}^d \rho_j(x)^{\beta_j - l - \sigma + |\alpha|} \prod_{k=1}^m \left( \frac{r_k(x)}{\rho(x)} \right)^{\delta_k - l - \sigma + |\alpha|} |\partial_x^\alpha u(x)| \\ &\quad + \sum_{|\alpha| = l} \sup_{|x-y| < r(x)/2} \prod_{j=1}^d \rho_j(x)^{\beta_j} \prod_{k=1}^m \left( \frac{r_k(x)}{\rho(x)} \right)^{\delta_k} \frac{|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)|}{|x-y|^\sigma}. \end{aligned}$$

Suppose that  $\delta_k \geq 0$  for  $k = 1, \dots, m$ . Then  $C_{\beta, \delta}^{l, \sigma}(\mathcal{G})$ , is defined as the set of all  $l$  times continuously differentiable functions on  $\bar{\mathcal{G}} \setminus \mathcal{S}$  with finite norm

$$\begin{aligned} \|u\|_{C_{\beta, \delta}^{l, \sigma}(\mathcal{G})} &= \sum_{|\alpha| \leq l} \sup_{x \in \mathcal{G}} \prod_{j=1}^d \rho_j(x)^{\beta_j - l - \sigma + |\alpha|} \prod_{k=1}^m \left( \frac{r_k(x)}{\rho(x)} \right)^{\max(0, \delta_k - l - \sigma + |\alpha|)} |\partial_x^\alpha u(x)| \\ &\quad + \sum_{j=1}^d \sum_{k: \sigma_k \leq l} \sum_{|\alpha| = l - \sigma_k} \sup_{\substack{x, y \in \mathcal{G}_{j, k} \\ |x-y| < \rho(x)/2}} \rho_j(x)^{\beta_j - \delta_k} \frac{|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)|}{|x-y|^{\sigma + \sigma_k - \delta_k}} \\ &\quad + \sum_{|\alpha| = l} \sup_{|x-y| < r(x)/2} \prod_{j=1}^d \rho_j(x)^{\beta_j} \prod_{k=1}^m \left( \frac{r_k(x)}{\rho(x)} \right)^{\delta_k} \frac{|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)|}{|x-y|^\sigma}, \end{aligned}$$

where  $\mathcal{G}_{j, k} = \{x \in \mathcal{G} : \rho_j(x) < 3\rho(x)/2, r_k(x) < 3r(x)/2\}$  and  $\sigma_k = [\delta_k - \sigma] + 1$ . The trace space on  $\Gamma_j$  for  $C_{\beta, \delta}^{l, \sigma}(\mathcal{G})$  is denoted by  $C_{\beta, \delta}^{l, \sigma}(\Gamma_j)$ .

Analogously to the case when the domain is a cone, we have

$$\mathcal{N}_{\beta, \delta}^{l, \sigma}(\mathcal{K}) \subset \mathcal{N}_{\beta', \delta'}^{l', \sigma'}(\mathcal{K}) \quad \text{if } l + \sigma \geq l' + \sigma', \beta_j - l - \sigma \leq \beta'_j - l' - \sigma', \delta_k - l - \sigma \leq \delta'_k - l' - \sigma'$$

for  $j = 1, \dots, d, k = 1, \dots, m$ . If in addition  $\delta_k$  and  $\delta'_k$  are nonnegative, then  $C_{\beta, \delta}^{l, \sigma}(\mathcal{G}) \subset C_{\beta', \delta'}^{l', \sigma'}(\mathcal{G})$ . Furthermore, it follows from Lemma 2.4 that

$$V_{\beta, \delta}^{l, s}(\mathcal{G}) \subset \mathcal{N}_{\beta', \delta'}^{l', \sigma}(\mathcal{G}) \quad \text{if } l - 3/s > l' + \sigma, \beta_j - l + 3/s \leq \beta'_j - l' - \sigma, \delta_k - l + 3/s \leq \delta'_k - l' - \sigma$$

for  $j = 1, \dots, d, k = 1, \dots, m$ .

We introduce the following notation. If  $\beta \in \mathbb{R}^d$ ,  $\delta \in \mathbb{R}^m$  and  $s, t \in \mathbb{R}$ , then by  $\mathcal{N}_{\beta+s, \delta+t}^{l, \sigma}(\mathcal{G})$  and  $C_{\beta+s, \delta+t}^{l, \sigma}(\mathcal{G})$ , we mean the spaces  $\mathcal{N}_{\beta', \delta'}^{l, \sigma}(\mathcal{G})$  and  $C_{\beta', \delta'}^{l, \sigma}(\mathcal{G})$  with  $\beta' = (\beta_1 + s, \dots, \beta_d + s)$ ,  $\delta' = (\delta_1 + t, \dots, \delta_m + t)$ . Analogous notation will be used for the weighted Sobolev spaces  $V_{\beta, \delta}^{l, s}$  and  $W_{\beta, \delta}^{l, s}$ .

The next lemma follows immediately from the definition of the space  $\mathcal{N}_{\beta, \delta}^{l, \sigma}(\mathcal{G})$ .

**Lemma 2.7** *If  $f \in \mathcal{N}_{\beta,\delta}^{l,\sigma}(\mathcal{G})$  and  $g \in \mathcal{N}_{\beta',\delta'}^{l,\sigma}(\mathcal{G})$ , then  $fg \in \mathcal{N}_{\beta+\beta'-l-\sigma,\delta+\delta'-l-\sigma}^{l,\sigma}(\mathcal{G})$ .*

Finally, we define  $C_{\beta,\delta}^{-1,\sigma}(\mathcal{G})$  as the space of all distributions of the form

$$f = f_0 + \sum_{j=1}^3 \partial_{x_j} f_j, \quad \text{where } f_0 \in C_{\beta+1,\delta+1}^{0,\sigma}(\mathcal{G}) \text{ and } f_j \in C_{\beta,\delta}^{0,\sigma}(\mathcal{G}), \quad j = 1, 2, 3. \quad (2.8)$$

Note that every  $f \in C^{-1,\sigma}(\mathcal{G})$ , i.e. every distribution of the form

$$f = f_0 + \sum_{j=1}^3 \partial_{x_j} f_j, \quad \text{where } f_j \in C^{0,\sigma}(\mathcal{G}), \quad j = 0, 1, 2, 3, \quad (2.9)$$

belongs to  $C_{0,0}^{-1,\sigma}(\mathcal{G})$ . Indeed, let  $\chi_k$  be infinitely differentiable cut-off functions equal to one near  $x^{(k)}$  and to zero near the vertices  $x^{(l)}$ ,  $l \neq k$ . Then the distribution (2.9) can be written as

$$f = F_0 + \sum_{j=1}^3 \partial_{x_j} F_j, \quad \text{where } F_0(x) = f_0(x) - \sum_{k=1}^d \sum_{j=1}^3 f_j(x^{(k)}) \partial_{x_j} \chi_k(x), \quad F_j(x) = f_j(x) - \sum_{k=1}^d \chi_k(x) f_j(x^{(k)}),$$

$j = 1, 2, 3$ . Here,  $F_0 \in C^{0,\sigma}(\mathcal{G}) \subset C_{1,1}^{0,\sigma}(\mathcal{G})$ , and from  $F_j \in C^{0,\sigma}(\mathcal{G})$ ,  $F_j(x^{(k)}) = 0$  it follows that  $F_j \in C_{0,0}^{0,\sigma}(\mathcal{G})$  for  $j = 1, 2, 3$ .

### 3 Regularity results for weak solutions

In this section, we establish regularity results for weak solutions in weighted Sobolev and Hölder spaces. The regularity assertions are formulated in terms of eigenvalues of operator pencils generated by the boundary value problem at the edge points and vertices of the domain.

#### 3.1 Operator pencils generated by the boundary value problem

We introduce the operator pencils generated by the problem (1.2), (1.3) for the edge points and vertices of the domain  $\mathcal{G}$ .

1) Let  $\xi$  be a point on an edge  $M_k$ , and let  $\Gamma_{k+}, \Gamma_{k-}$  be the faces of  $\mathcal{G}$  adjacent to  $\xi$ . Then by  $\mathcal{D}_\xi$  we denote the dihedron which is bounded by the half-planes  $\Gamma_{k\pm}^\circ$  tangent to  $\Gamma_{k\pm}$  at  $\xi$  and the edge  $M_\xi^\circ = \bar{\Gamma}_{k+}^\circ \cap \bar{\Gamma}_{k-}^\circ$ . The angle between the half-planes  $\Gamma_{k\pm}^\circ$  is denoted by  $\theta_\xi$ . Furthermore, let  $r, \varphi$  be polar coordinates in the plane perpendicular to  $M_\xi^\circ$  such that

$$\Gamma_{k\pm}^\circ = \{x \in \mathbb{R}^3 : r > 0, \varphi = \pm\theta_\xi/2\}.$$

Then we define the operator  $A_\xi(\lambda)$  as follows:

$$A_\xi(\lambda) (U(\varphi), P(\varphi)) = (r^{2-\lambda}(-\Delta u + \nabla p), -r^{1-\lambda} \nabla \cdot u, r^{-\lambda} S_{k\pm} u|_{\varphi=\pm\theta_\xi/2}, r^{1-\lambda} N_{k\pm}(u, p)|_{\varphi=\pm\theta_\xi/2}),$$

where  $u(x) = r^\lambda U(\varphi)$ ,  $p(x) = r^{\lambda-1} P(\varphi)$ ,  $\lambda \in \mathbb{C}$ . The operator  $A_\xi(\lambda)$  depends quadratically on the parameter  $\lambda$  and realizes a continuous mapping

$$W^{2,2}(I_\xi)^3 \times W^{1,2}(I_\xi) \rightarrow W^{1,2}(I_\xi)^3 \times L_2(I_\xi) \times \mathbb{C}^3 \times \mathbb{C}^3$$

for every  $\lambda \in \mathbb{C}$ , where  $I_\xi$  denotes the interval  $(-\theta_\xi/2, +\theta_\xi/2)$ . The spectrum of the pencil  $A_\xi(\lambda)$  consists of eigenvalues with finite geometric and algebraic multiplicities. These eigenvalues are zeros of certain transcendental functions (see [17]). For example, in the cases  $d_{k+} = d_{k-} = 0$  (Dirichlet conditions on  $\Gamma_{k\pm}$ ) and  $d_{k+} = d_{k-} = 3$  (Neumann conditions on  $\Gamma_{k\pm}$ ), the spectrum of  $A_\xi(\lambda)$  consists of the solutions of the equation

$$\sin(\lambda\theta_\xi) (\lambda^2 \sin^2 \theta_\xi - \sin^2(\lambda\theta_\xi)) = 0,$$

$\lambda \neq 0$  for  $d_{k+} = d_{k-} = 0$ .

Let  $\lambda_1(\xi)$  be the eigenvalue with smallest positive real part of this pencil, and let  $\lambda_2(\xi)$  be the eigenvalue with smallest real part greater than 1. We define

$$\mu(\xi) = \begin{cases} \operatorname{Re} \lambda_2(\xi) & \text{if } d_{k+} + d_{k-} \text{ is even and } \theta_\xi < \pi/m_k, \\ \operatorname{Re} \lambda_1(\xi) & \text{else,} \end{cases}$$

where  $m_k = 1$  if  $d_{k+} + d_{k-} \in \{0, 6\}$ ,  $m_k = 2$  if  $d_{k+} + d_{k-} \in \{2, 4\}$ . Finally, let

$$\mu_k = \inf_{\xi \in M_k} \mu(\xi). \quad (3.1)$$

Note that in the case of even  $d_{k+} + d_{k-}$ , the number  $\lambda = 1$  belongs always to the spectrum of the pencil  $A_\xi(\lambda)$ .

2) Let  $x^{(j)}$  be a vertex of  $\mathcal{G}$  and let  $I_j$  be the set of all indices  $k$  such that  $x^{(j)} \in \bar{\Gamma}_k$ . By our assumptions, there exist a neighborhood  $\mathcal{U}$  of  $x^{(j)}$  and a diffeomorphism  $\kappa$  mapping  $\mathcal{G} \cap \mathcal{U}$  onto  $\mathcal{K}_j \cap B_1$  and  $\Gamma_k \cap \mathcal{U}$  onto  $\Gamma_k^\circ \cap B_1$  for  $k \in I_j$ , where  $\mathcal{K}_j = \{x : x/|x| \in \Omega_j\}$  is a polyhedral cone with vertex 0 and  $\Gamma_k^\circ = \{x : x/|x| \in \gamma_k\}$  are the faces of this cone. Without loss of generality, we may assume that the Jacobian matrix  $\kappa'(x)$  is equal to the identity matrix  $I$  at the point  $x^{(j)}$ . We introduce spherical coordinates  $\rho = |x|$ ,  $\omega = x/|x|$  in  $\mathcal{K}_j$  and define

$$V_{\Omega_j} = \{u \in W^{1,2}(\Omega_j)^3 : S_k u = 0 \text{ on } \gamma_k, k \in I_j\}.$$

On the space  $V_{\Omega_j} \times L_2(\Omega_j)$ , we define the bilinear form  $a_j(\cdot, \cdot; \lambda)$  as

$$a_j\left(\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix}; \lambda\right) = \frac{1}{\log 2} \int_{\substack{\mathcal{K} \\ 1 < |x| < 2}} \left(2\nu \sum_{i,j=1}^3 \varepsilon_{i,j}(U) \cdot \varepsilon_{i,j}(V) - P \nabla \cdot V - (\nabla \cdot U) Q\right) dx,$$

where  $U = \rho^\lambda u(\omega)$ ,  $V = \rho^{-1-\lambda} v(\omega)$ ,  $P = \rho^{\lambda-1} p(\omega)$ ,  $Q = \rho^{-2-\lambda} q(\omega)$ ,  $u, v \in V_{\Omega_j}$ ,  $p, q \in L_2(\Omega_j)$ , and  $\lambda \in \mathbb{C}$ . This bilinear form generates the linear and continuous operator

$$\mathfrak{A}_j(\lambda) : V_{\Omega_j} \times L_2(\Omega_j) \rightarrow V_{\Omega_j}^* \times L_2(\Omega_j)$$

by

$$\int_{\Omega} \mathfrak{A}_j(\lambda) \begin{pmatrix} u \\ p \end{pmatrix} \cdot \begin{pmatrix} v \\ q \end{pmatrix} d\omega = a_j\left(\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix}; \lambda\right), \quad u, v \in V_{\Omega_j}, \quad p, q \in L_2(\Omega_j).$$

The operator  $\mathfrak{A}_j(\lambda)$  depends quadratically on the complex parameter  $\lambda$ . The spectrum of the pencil  $\mathfrak{A}_j(\lambda)$  consists of isolated points, eigenvalues with finite geometric and algebraic multiplicities.

### 3.2 Regularity assertions for weak solutions of the linearized problem

The following two theorems are proved in [18].

**Theorem 3.1** *Let  $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$  be a solution of the problem (1.9), (1.10). Suppose that the following conditions are satisfied.*

- (i)  $F \in V^* \cap V_{\beta,\delta}^{-1,s}(\mathcal{G})^3$ ,  $g \in L_2(\mathcal{G}) \cap W_{\beta,\delta}^{0,s}(\mathcal{G})$ ,  $h_j \in W^{1/2,2}(\Gamma_j) \cap W_{\beta,\delta}^{1-1/s,s}(\Gamma_j)$ ,
- (ii) *there are no eigenvalues of the pencils  $\mathfrak{A}_j(\lambda)$ ,  $j = 1, \dots, d$  in the closed strip between the lines  $\operatorname{Re} \lambda = -1/2$  and  $\operatorname{Re} \lambda = 1 - \beta_j - 3/s$ ,*
- (iii) *the components of  $\delta$  satisfy the inequalities  $\max(1 - \mu_k, 0) < \delta_k + 2/s < 1$ .*

*Then  $u \in W_{\beta,\delta}^{1,s}(\mathcal{G})^3$  and  $p \in W_{\beta,\delta}^{0,s}(\mathcal{G})$ .*

**Theorem 3.2** *Let  $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$  be a solution of the problem (1.9), (1.10). Suppose that*

- (i)  $g \in W_{\beta,\delta}^{1,s}(\mathcal{G})$ ,  $h_j \in W_{\beta,\delta}^{2-1/s,s}(\Gamma_j)^{3-d_j}$ , and  $F \in V^*$  has the representation (1.7) with  $f \in W_{\beta,\delta}^{0,s}(\mathcal{G})^3$ ,  $\phi_j \in W_{\beta,\delta}^{1-1/s}(\Gamma_j)^{d_j}$ ,
- (ii) there are no eigenvalues of the pencils  $\mathfrak{A}_j(\lambda)$ ,  $j = 1, \dots, d$ , in the closed strip between the lines  $\operatorname{Re} \lambda = -1/2$  and  $\operatorname{Re} \lambda = 2 - \beta_j - 3/s$ ,
- (iii) the components of  $\delta$  satisfy the inequalities  $\max(2 - \mu_k, 0) < \delta_k + 2/s < 2$ ,
- (iv)  $g$ ,  $h_j$  and  $\phi_j$  are such that there exist  $w \in W_{\beta,\delta}^{2,s}(\mathcal{G})^3$  and  $q \in W_{\beta,\delta}^{1,s}(\mathcal{G})$  satisfying

$$S_j w = h_j, \quad N_j(w, q) = \phi_j \quad \text{on } \Gamma_j, \quad j = 1, \dots, n, \quad \nabla \cdot w + g \in V_{\beta,\delta}^{1,s}(\mathcal{G}).$$

Then  $u \in W_{\beta,\delta}^{2,s}(\mathcal{G})^3$  and  $p \in W_{\beta,\delta}^{1,s}(\mathcal{G})$ .

For the proof of the following two regularity assertions in weighted Hölder spaces, we refer to [19].

**Theorem 3.3** Let  $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$  be a weak solution of the problem (1.8), (1.3). Suppose that

- (i)  $f \in C_{\beta,\delta}^{-1,\sigma}(\mathcal{G})^3$ ,  $g \in C_{\beta,\delta}^{0,\sigma}(\mathcal{G})$ ,  $h_j \in C_{\beta,\delta}^{1,\sigma}(\Gamma_j)^{3-d_j}$ ,  $\phi_j \in C_{\beta,\delta}^{0,\sigma}(\Gamma_j)^{d_j}$ ,
- (ii)  $\beta_j - \sigma < 3/2$  for  $j = 1, \dots, d$ , and the strip  $-1/2 < \operatorname{Re} \lambda \leq 1 + \sigma - \beta_j$  is free of eigenvalues of the pencils  $\mathfrak{A}_j(\lambda)$ ,  $j = 1, \dots, d$ ,
- (iii) the components of  $\delta$  are nonnegative and satisfy the inequalities  $1 - \mu_k < \delta_k - \sigma < 1$ ,  $\delta_k \neq \sigma$ ,
- (iv)  $g$ ,  $h_j$  and  $\phi_j$  are such that there exist  $w \in C_{\beta,\delta}^{1,\sigma}(\mathcal{G})^3$  and  $q \in C_{\beta,\delta}^{1,\sigma}(\mathcal{G})$  satisfying

$$S_j w = h_j, \quad N_j(w, q) = \phi_j \quad \text{on } \Gamma_j, \quad j = 1, \dots, n, \quad \nabla \cdot w + g \in \mathcal{N}_{\beta,\delta}^{0,\sigma}(\mathcal{G}).$$

Then  $(u, p) \in C_{\beta,\delta}^{1,\sigma}(\mathcal{G})^3 \times C_{\beta,\delta}^{0,\sigma}(\mathcal{G})$ .

Note that under the conditions of Theorem 3.3 on  $\beta$  and  $\delta$ , there are the imbeddings  $C_{\beta,\delta}^{-1,\sigma}(\mathcal{G})^3 \subset V^*$ ,  $C_{\beta,\delta}^{0,\sigma}(\mathcal{G}) \subset L_2(\mathcal{G})$ , and  $C_{\beta,\delta}^{1,\sigma}(\Gamma_j) \subset W^{1/2,2}(\mathcal{G})$ .

**Theorem 3.4** Let  $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$  be a weak solution of the problem (1.8), (1.3). Suppose that

- (i)  $f \in C_{\beta,\delta}^{0,\sigma}(\mathcal{G})^3$ ,  $g \in C_{\beta,\delta}^{1,\sigma}(\mathcal{G})$ ,  $h_j \in C_{\beta,\delta}^{2,\sigma}(\Gamma_j)^{3-d_j}$ ,  $\phi_j \in C_{\beta,\delta}^{1,\sigma}(\Gamma_j)^{d_j}$ ,
- (ii)  $\beta_j - \sigma < 5/2$  for  $j = 1, \dots, d$ , the strip  $-1/2 < \operatorname{Re} \lambda \leq 2 + \sigma - \beta_j$  is free of eigenvalues the pencil  $\mathfrak{A}_j(\lambda)$ ,  $j = 1, \dots, d$ ,
- (iii) the components of  $\delta$  are nonnegative, satisfy the inequalities  $2 - \mu_k < \delta_k - \sigma < 2$ ,  $\delta_k \neq \sigma$ , and  $\delta_k \neq 1 + \sigma$ ,
- (iv)  $g$ ,  $h_j$  and  $\phi_j$  are such that there exist  $w \in C_{\beta,\delta}^{2,\sigma}(\mathcal{G})^3$  and  $q \in C_{\beta,\delta}^{1,\sigma}(\mathcal{G})$  satisfying

$$S_j w = h_j, \quad N_j(w, q) = \phi_j \quad \text{on } \Gamma_j, \quad j = 1, \dots, n, \quad \nabla \cdot w + g \in \mathcal{N}_{\beta,\delta}^{1,\sigma}(\mathcal{G}).$$

Then  $(u, p) \in C_{\beta,\delta}^{2,\sigma}(\mathcal{G})^3 \times C_{\beta,\delta}^{1,\sigma}(\mathcal{G})$ .

**Remark 3.1** For the validity of condition (iv) in Theorems 3.2–3.4 it is necessary and sufficient that the functions  $h_j$  and their derivatives,  $\phi_j$  and  $g$  satisfy certain compatibility conditions on the edges of the domain  $\mathcal{G}$  (see [18, 19]).

### 3.3 Regularity results for solutions of the nonlinear problem in weighted Sobolev spaces

Our goal is to extend the results of Theorems 3.1–3.4 to the nonlinear problem (1.2), (1.3). We start with regularity results in weighted Sobolev spaces.

**Lemma 3.1** *Let  $u \in L_6(\mathcal{G}) \cap W_{\beta,\delta}^{1,s}(\mathcal{G})$ ,  $s > 6/5$ ,  $\beta'_j \geq \beta_j - 1/2$  for  $j = 1, \dots, d$ , and  $\delta'_k \geq \delta_k - 1/2$ . Then  $u \partial_{x_i} u \in V_{\beta',\delta'}^{-1,s}(\mathcal{G})$  for  $i = 1, 2, 3$ .*

*P r o o f.* Let  $q = 6s/(s+6)$ . By Hölders inequality,

$$\|u \partial_{x_i} u\|_{V_{\beta,\delta}^{0,q}(\mathcal{G})} \leq \|u\|_{L_6(\mathcal{G})} \|\partial_{x_i} u\|_{V_{\beta,\delta}^{0,s}(\mathcal{G})}.$$

Furthermore by Corollary 2.2, the space  $V_{\beta,\delta}^{0,q}(\mathcal{G})$  is continuously imbedded into  $V_{\beta',\delta'}^{-1,s}(\mathcal{G})$  if  $\beta'_j \geq \beta_j - 1/2$  and  $\delta'_k \geq \delta_k - 1/2$ . The result follows.  $\square$

**Theorem 3.5** *Let  $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$  be a solution of the problem (1.4), (1.5). We suppose that  $s > 6/5$  and that the conditions (i)–(iii) of Theorem 3.1 are satisfied. Then  $u \in W_{\beta,\delta}^{1,s}(\mathcal{G})^3$  and  $p \in W_{\beta,\delta}^{0,s}(\mathcal{G})$ .*

*P r o o f.* 1) First, let  $s \leq 3$ . From  $u_j \in W^{1,2}(\mathcal{G}) \subset L_6(\mathcal{G})$  and  $\partial_{x_j} u \in L_2(\mathcal{G})^3$  it follows that  $u_j \partial_{x_j} u \in L_{3/2}(\mathcal{G})^3$ . This together with Corollary 2.2 implies  $(u \cdot \nabla) u \in V_{1-3/s, 1-3/s}^{-1,s}(\mathcal{G})^3$ . Hence,  $(u, p)$  is a solution of the problem

$$b(u, v) - \int_{\mathcal{G}} p \nabla \cdot v \, dx = \Phi(v) \quad \text{for all } v \in V, \quad (3.2)$$

$$-\nabla \cdot u = g \quad \text{in } \mathcal{G}, \quad S_j u = h_j \quad \text{on } \Gamma_j, \quad j = 1, \dots, N, \quad (3.3)$$

where

$$\Phi = F - (u \cdot \nabla) u \in V_{\beta',\delta'}^{-1,s}(\mathcal{G})^3, \quad \beta'_j = \max(\beta_j, 1 - 3/s), \quad \delta'_k = \max(\delta_k, 1 - 3/s).$$

From Theorem 3.1 we conclude that  $(u, p) \in W_{\beta',\delta'}^{1,s}(\mathcal{G})^3 \times W_{\beta',\delta'}^{0,s}(\mathcal{G})$ . Then by Lemma 3.1, we have  $(u \cdot \nabla) u \in V_{\beta'-1/2, \delta'-1/2}^{-1,s}(\mathcal{G})^3$  and therefore,

$$F - (u \cdot \nabla) u \in V_{\beta'',\delta''}^{-1,s}(\mathcal{G})^3, \quad \text{where } \beta''_j = \max(\beta_j, 1/2 - 3/s), \quad \delta''_k = \max(\delta_k, 1/2 - 3/s).$$

Consequently, Theorem 3.1 implies  $(u, p) \in W_{\beta'',\delta''}^{1,s}(\mathcal{G})^3 \times W_{\beta'',\delta''}^{0,s}(\mathcal{G})$ . Repeating the last consideration, we obtain  $(u, p) \in W_{\beta,\delta}^{1,s}(\mathcal{G})^3 \times W_{\beta,\delta}^{0,s}(\mathcal{G})$ .

2) Next, let  $3 < s \leq 6$ . Then by Lemma 2.5,  $V_{\beta,\delta}^{-1,s}(\mathcal{G}) \subset V_{\beta',\delta'}^{-1,3}(\mathcal{G})$ ,  $W_{\beta,\delta}^{0,s}(\mathcal{G}) \subset W_{\beta',\delta'}^{0,3}(\mathcal{G})$ , and  $W_{\beta,\delta}^{1-1/s,s}(\Gamma_j) \subset W_{\beta',\delta'}^{2/3,3}(\Gamma_j)$ , where  $\beta'_j = \beta + 3/s - 1 + \varepsilon$ ,  $\delta'_k = \delta_k + 2/s - 2/3 + \varepsilon$ ,  $\varepsilon$  is an arbitrarily small positive number. For sufficiently small  $\varepsilon$ , we conclude from part 1) that  $u \in W_{\beta',\delta'}^{1,3}(\mathcal{G})^3$ . By Hölders inequality,

$$\|u_j \partial_{x_j} u\|_{W_{\beta',\delta'}^{0,2}(\mathcal{G})^3} \leq \|u_j\|_{L_6(\mathcal{G})} \|\partial_{x_j} u\|_{W_{\beta',\delta'}^{0,3}(\mathcal{G})^3}$$

Due to Corollary 2.2,  $W_{\beta',\delta'}^{0,2}(\mathcal{G}) \subset V_{\beta,\delta}^{-1,s}(\mathcal{G})$  if  $\varepsilon < 1/3$ . Therefore,  $(u, p)$  is a solution of the problem (3.2), (3.3), where  $\Phi = F - (u \cdot \nabla) u \in V_{\beta,\delta}^{-1,s}(\mathcal{G})^3$ . Using Theorem 3.1, we obtain  $(u, p) \in W_{\beta,\delta}^{1,s}(\mathcal{G})^3 \times W_{\beta,\delta}^{0,s}(\mathcal{G})$ .

3) Finally, let  $s > 6$ . Then again by Lemma 2.5,  $V_{\beta,\delta}^{-1,s}(\mathcal{G}) \subset V_{\beta',\delta'}^{-1,6}(\mathcal{G})$ ,  $W_{\beta,\delta}^{0,s}(\mathcal{G}) \subset W_{\beta',\delta'}^{0,6}(\mathcal{G})$ , and  $W_{\beta,\delta}^{1-1/s,s}(\Gamma_j) \subset W_{\beta',\delta'}^{5/6,6}(\Gamma_j)$ , where  $\beta'_j = \beta + 3/s - 1/2 + \varepsilon$ ,  $\delta'_k = \delta_k + 2/s - 1/3 + \varepsilon$ ,  $\varepsilon$  is an arbitrarily small positive number. For sufficiently small  $\varepsilon$ , we conclude from part 2) that  $u \in W_{\beta',\delta'}^{1,6}(\mathcal{G})^3$ . Since  $u \in L_6(\mathcal{G})^3$ , it follows that  $(u \cdot \nabla) u \in W_{\beta',\delta'}^{0,3}(\mathcal{G})^3$ . The last space is embedded to  $V_{\beta,\delta}^{-1,s}(\mathcal{G})^3$  if  $\varepsilon \leq 1/3$  (see Corollary 2.2). Therefore,  $(u, p)$  is a solution of the problem (3.2), (3.3), where  $\Phi \in V_{\beta,\delta}^{-1,s}(\mathcal{G})^3$ . Applying Theorem 3.1, we obtain  $(u, p) \in W_{\beta,\delta}^{1,s}(\mathcal{G})^3 \times W_{\beta,\delta}^{0,s}(\mathcal{G})$ .  $\square$

For the proof of the analogous  $W_{\beta,\delta}^{2,s}$  regularity result, we need the following lemma.

**Lemma 3.2** *Let  $u \in W_{\beta,\delta}^{2,s}(\mathcal{G}) \cap W^{1,2}(\mathcal{G})$ ,  $1 < s < 6$ ,  $\beta_j + 3/s \leq 5/2$ ,  $\delta_k + 2/s > 0$ . Then  $u \nabla u \in V_{\beta-1/2,\delta'}^{0,s}(\mathcal{G})^3$  for every  $\delta'$ ,  $\delta'_k \geq \delta_k - 1/2$ ,  $\delta'_k + 2/s > 0$ .*

*P r o o f.* 1) Suppose that  $\delta_k + 2/s > 1$  for  $k = 1, \dots, m$ . Then by Lemma 2.6,  $\nabla u \in W_{\beta,\delta}^{1,s}(\mathcal{G})^3 = V_{\beta,\delta}^{1,s}(\mathcal{G})^3 \subset V_{\beta-1/2,\delta-1/2}^{0,q}(\mathcal{G})^3$ ,  $q = 6s/(6-s)$ . From this, from the assumption  $u \in W^{1,2}(\mathcal{G}) \subset L_6(\mathcal{G})$  and from Hölder's inequality it follows that  $u \nabla u \in V_{\beta-1/2,\delta-1/2}^{0,s}(\mathcal{G})^3$ ,

$$\|u \nabla u\|_{V_{\beta-1/2,\delta-1/2}^{0,s}(\mathcal{G})^3} \leq c \|u\|_{L_6(\mathcal{G})} \|\nabla u\|_{V_{\beta-1/2,\delta-1/2}^{0,q}(\mathcal{G})^3}$$

2) We consider the case when  $0 < \delta_k + 2/s \leq 1$  for all  $k$ . Then  $u$  admits the decomposition

$$u = v + w, \quad v \in V_{\beta,\delta}^{2,s}(\mathcal{G}), \quad w \in W_{\beta+2,\delta+2}^{4,s}(\mathcal{G}).$$

From Lemma 2.2 it follows that

$$\prod_j \rho_j^{\beta_j-1+2/s} \nabla w \in L_{3s}(\mathcal{G})^3, \quad \prod_j \rho_j^{\beta_j-2+3/s} w \in L_\infty(\mathcal{G}), \quad \prod_j \rho_j^{\beta_j-5/2+3/s} w \in L_6(\mathcal{G}). \quad (3.4)$$

Since  $\beta_j - 5/2 + 3/2 \leq 0$ , we have in particular  $w \in L_6(\mathcal{G})$  and therefore also  $v \in L_6(\mathcal{G})$ . We estimate the norms of  $v \nabla u$  and  $w \nabla u$  in  $V_{\beta-1/2,\delta-1/2}^{0,s}(\mathcal{G})^3$ . Let  $q = 6s/(6-s)$ . Using Hölder's inequality and Lemma 2.6, we obtain

$$\|v \nabla v\|_{V_{\beta-1/2,\delta-1/2}^{0,s}(\mathcal{G})^3} \leq \|v\|_{L_6(\mathcal{G})} \|\nabla v\|_{V_{\beta-1/2,\delta-1/2}^{0,q}(\mathcal{G})^3} \leq c \|v\|_{L_6(\mathcal{G})} \|\nabla v\|_{V_{\beta,\delta}^{1,s}(\mathcal{G})^3}.$$

By Lemma 2.6, the space  $V_{\beta,\delta}^{2,s}(\mathcal{G})$  is continuously imbedded into  $V_{\beta-2+1/s,\delta-2+1/s}^{0,3s/2}(\mathcal{G})$ . Consequently,

$$\begin{aligned} \|v \nabla w\|_{V_{\beta-1/2,\delta-1/2}^{0,s}(\mathcal{G})^3} &\leq \|v\|_{V_{\beta-2+1/s,\delta-2+1/s}^{0,3s/2}(\mathcal{G})} \left\| \prod_j \rho_j^{3/2-1/s} \prod_k \left(\frac{r_k}{\rho}\right)^{3/2-1/s} \nabla w \right\|_{L_{3s}(\mathcal{G})^3} \\ &\leq c \|v\|_{V_{\beta,\delta}^{2,s}(\mathcal{G})} \left\| \prod_j \rho_j^{\beta_j-1+2/s} \nabla w \right\|_{L_{3s}(\mathcal{G})^3}. \end{aligned}$$

Thus, we have  $v \nabla u \in V_{\beta-1/2,\delta-1/2}^{0,s}(\mathcal{G})^3 \subset V_{\beta-1/2,\delta'}^{0,s}(\mathcal{G})^3$ . Furthermore, using the continuity of the imbedding  $W_{\beta,\delta}^{2,s}(\mathcal{G}) \subset W_{3/2-3/s,\delta'}^{1,s}(\mathcal{G})$ , we obtain

$$\begin{aligned} \|w \nabla u\|_{V_{\beta-1/2,\delta'}^{0,s}(\mathcal{G})^3} &\leq \left\| \prod_j \rho_j^{\beta_j-2+3/s} w \right\|_{L_\infty(\mathcal{G})} \left\| \prod_j \rho_j^{3/2-3/s} \prod_k \left(\frac{r_k}{\rho}\right)^{\delta'_k} \nabla u \right\|_{L_s(\mathcal{G})^3} \\ &\leq c \left\| \prod_j \rho_j^{\beta_j-2+3/s} w \right\|_{L_\infty(\mathcal{G})} \|u\|_{W_{\beta,\delta}^{2,s}(\mathcal{G})^3}. \end{aligned}$$

This proves the lemma for the case  $\delta_k + 2/s < 1$ ,  $k = 1, \dots, m$ .

3) The case when  $\delta_k + 2/s < 1$  for some but not all  $k$  can be reduced to cases 1) and 2) using suitable cut-off functions.  $\square$

**Theorem 3.6** *Let  $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$  be a solution of the problem (1.4), (1.5). We assume that the conditions (i)–(iv) of Theorem 3.2 are satisfied and that  $\beta_j + 3/s < 5/2$  for  $j = 1, \dots, d$ . Then  $u \in W_{\beta,\delta}^{2,s}(\mathcal{G})^3$  and  $p \in W_{\beta,\delta}^{1,s}(\mathcal{G})$ .*

*P r o o f.* 1) Let first  $1 < s \leq 3/2$ . We put  $q = 3s/(3-2s)$  if  $s < 3/2$ ,  $q = \infty$  if  $s = 3/2$ . Since

$$\|u_i \partial_{x_i} u\|_{W_{\beta',\delta'}^{0,s}(\mathcal{G})^3} \leq \|u_i\|_{L_6(\mathcal{G})} \|\partial_{x_i} u\|_{L_2(\mathcal{G})^3} \left\| \prod_j \rho_j^{\beta'_j} \prod_k \left(\frac{r_k}{\rho}\right)^{\delta'_k} \right\|_{L_q(\mathcal{G})},$$



we obtain  $(u \cdot \nabla)u \in W_{\beta', \delta'}^{0,s}(\mathcal{G})^3$  if  $\beta'_j > -3/q$  and  $\delta'_k > -2/q$  or, what is the same, if  $\beta'_j + 3/s > 2$  for  $j = 1, \dots, N$ ,  $\delta'_k + 2/s > 4/3$  for  $k = 1, \dots, m$ . Let  $\beta'_j = \max(\beta_j, 2 - 3/s + \varepsilon)$ ,  $\delta'_k = \max(\delta_k, 4/3 - 2/s + \varepsilon)$ , where  $\varepsilon$  is a sufficiently small positive number. Then  $(u, p)$  is a solution of the problem

$$-\nu \Delta u + \nabla p = f', \quad -\nabla \cdot u = g \quad \text{in } \mathcal{G} \quad (3.5)$$

$$S_j u = h_j, \quad N_j(u, p) = \phi_j \quad \text{on } \Gamma_j, \quad j = 1, \dots, N, \quad (3.6)$$

where  $f' = f - (u \cdot \nabla)u \in W_{\beta', \delta'}^{0,s}(\mathcal{G})^3$ ,  $g \in W_{\beta', \delta'}^{1,s}(\mathcal{G})$ ,  $h_j \in W_{\beta', \delta'}^{2-1/s, s}(\Gamma_j)$  and  $\phi_j \in W_{\beta', \delta'}^{1-1/s, s}(\Gamma_j)$ . Consequently by Theorem 3.2, we have  $(u, p) \in W_{\beta', \delta'}^{2,s}(\mathcal{G})^3 \times W_{\beta', \delta'}^{1,s}(\mathcal{G})$ . Applying Lemma 3.2, we obtain  $f' \in W_{\beta'', \delta''}^{0,s}(\mathcal{G})^3$ , where  $\beta''_j = \max(\beta_j, 3/2 - 3/s + \varepsilon)$  and  $\delta''_k = \max(\delta_k, 5/6 - 2/s + \varepsilon)$ . Hence, Theorem 3.2 implies  $(u, p) \in W_{\beta'', \delta''}^{2,s}(\mathcal{G})^3 \times W_{\beta'', \delta''}^{1,s}(\mathcal{G})$ . Repeating this procedure, we obtain  $(u, p) \in W_{\beta, \delta}^{2,s}(\mathcal{G})^3 \times W_{\beta, \delta}^{1,s}(\mathcal{G})$ .

2) Next, we consider the case  $3/2 < s \leq 2$ . Let  $\varepsilon$  be a positive number less than  $1/2$  such that  $\delta_k + 2/s < 2 - \varepsilon$  for all  $k$ . Then by Lemma 2.5,  $W_{\beta, \delta}^{l,s}(\mathcal{G}) \subset W_{\beta+3/s-2+\varepsilon, 2/3-\varepsilon}^{l, 3/2}(\mathcal{G})$ , and from part 1) it follows that  $(u, p) \in W_{\beta+3/s-2+\varepsilon, 2/3-\varepsilon}^{2, 3/2}(\mathcal{G})^3 \times W_{\beta+3/s-2+\varepsilon, 2/3-\varepsilon}^{1, 3/2}(\mathcal{G})$  if  $\varepsilon$  is sufficiently small. In particular,

$$\partial_{x_i} u \in W_{\beta+3/s-2+\varepsilon, 2/3-\varepsilon}^{1, 3/2}(\mathcal{G})^3 = V_{\beta+3/s-2+\varepsilon, 2/3-\varepsilon}^{1, 3/2}(\mathcal{G})^3 \subset V_{\beta+3/s-2+\varepsilon, 2/3-\varepsilon}^{0, 3}(\mathcal{G})^3$$

(see Lemma 2.6). Let  $\delta'_k = \max(\delta_k, 5/3 - 2/s)$  for  $k = 1, \dots, m$ . Then

$$\|u_i \partial_{x_i} u\|_{W_{\beta, \delta'}^{0,s}(\mathcal{G})^3} \leq c \|u_i\|_{L_6(\mathcal{G})} \left\| \prod_j \rho_j^{\beta_j+3/s-2+\varepsilon} \prod_k \left(\frac{r_k}{\rho}\right)^{2/3-\varepsilon} \partial_{x_i} u \right\|_{L_3(\mathcal{G})^3},$$

where

$$c = \left\| \prod_j \rho_j^{2-3/s-\varepsilon} \prod_k \left(\frac{r_k}{\rho}\right)^{\delta'_k-2/3+\varepsilon} \right\|_{L_{2s/(2-s)}(\mathcal{G})} < \infty.$$

Consequently,  $f' = f - (u \cdot \nabla)u \in W_{\beta, \delta'}^{0,s}(\mathcal{G})^3$ , and Theorem 3.2 implies  $(u, p) \in W_{\beta, \delta'}^{2,s}(\mathcal{G})^3 \times W_{\beta, \delta'}^{1,s}(\mathcal{G})$ . Using Lemma 3.2, we obtain  $(u, p) \in W_{\beta, \delta}^{2,s}(\mathcal{G})^3 \times W_{\beta, \delta}^{1,s}(\mathcal{G})$  analogously to the first part of the proof.

3) Let  $2 < s \leq 3$ , and let  $\varepsilon$  be a positive number less than  $1/2$  such that  $\delta_k + 2/s < 2 - \varepsilon$  for all  $k$ . Then by Lemma 2.5,  $W_{\beta, \delta}^{l,s}(\mathcal{G}) \subset W_{\beta+3/s-3/2+\varepsilon, 1-\varepsilon}^{l, 2}(\mathcal{G})$ . Therefore by part 2), we have  $(u, p) \in W_{\beta+3/s-3/2+\varepsilon, 1-\varepsilon}^{2, 2}(\mathcal{G})^3 \times W_{\beta+3/s-3/2+\varepsilon, 1-\varepsilon}^{1, 2}(\mathcal{G})$  provided  $\varepsilon$  is sufficiently small. Consequently,

$$\partial_{x_i} u \in W_{\beta+3/s-3/2+\varepsilon, 1-\varepsilon}^{1, 2}(\mathcal{G})^3 = V_{\beta+3/s-3/2+\varepsilon, 1-\varepsilon}^{1, 2}(\mathcal{G})^3 \subset V_{\beta+3/s-3/2+\varepsilon, 1-\varepsilon}^{0, 6}(\mathcal{G})^3.$$

Let  $\delta'_k = \max(\delta_k, 5/3 - 2/s)$  for  $k = 1, \dots, m$ . Then

$$\|u_i \partial_{x_i} u\|_{W_{\beta, \delta'}^{0,s}(\mathcal{G})^3} \leq c \|u_i\|_{L_6(\mathcal{G})} \|\partial_{x_i} u\|_{V_{\beta+3/s-3/2+\varepsilon, 1-\varepsilon}^{0, 6}(\mathcal{G})^3},$$

where

$$c = \left\| \prod_j \rho_j^{3/2-3/s-\varepsilon} \prod_k \left(\frac{r_k}{\rho}\right)^{\delta'_k-1+\varepsilon} \right\|_{L_{3s/(3-s)}(\mathcal{G})} < \infty.$$

Thus, we have  $f' = f - (u \cdot \nabla)u \in W_{\beta, \delta'}^{0,s}(\mathcal{G})^3$  and  $(u, p) \in W_{\beta, \delta'}^{2,s}(\mathcal{G})^3 \times W_{\beta, \delta'}^{1,s}(\mathcal{G})$  (by Theorem 3.2). Using again Lemma 3.2, we obtain  $(u, p) \in W_{\beta, \delta}^{2,s}(\mathcal{G})^3 \times W_{\beta, \delta}^{1,s}(\mathcal{G})$ .

4) Finally, let  $s > 3$ . We define  $\delta'_k = \max(\delta_k, 1 - 2/s + \varepsilon)$ , where  $\varepsilon$  is a sufficiently small positive number. Then we have

$$g \in W_{\beta, \delta'}^{1,s}(\mathcal{G}) \subset W_{\beta-1, \delta'-1}^{0,s}(\mathcal{G}), \quad h_j \in W_{\beta, \delta'}^{2-1/s, s}(\Gamma_j)^{3-d_j} \subset W_{\beta-1, \delta'-1}^{1-1/s, s}(\Gamma_j)^{3-d_j}.$$

Furthermore, the functional (1.7) belongs to  $V_{\beta-1, \delta'-1}^{-1,s}(\mathcal{G})^3$ . Since  $\max(1 - \mu_k, 0) < \delta'_k - 1 + 2/s < 1$  it follows from Theorem 3.5 that  $u \in W_{\beta-1, \delta'-1}^{1,s}(\mathcal{G})^3$ . Then by Corollary 2.1,

$$\prod_j \rho_j^{\beta_j-2+3/s} \prod_k \left(\frac{r_k}{\rho}\right)^{\sigma_k} u \in L_\infty(\mathcal{G})^3,$$

where  $\sigma_k = 0$  for  $\delta_k < 2 - 3/s$ ,  $\sigma_k = 1/s + \varepsilon$  for  $2 - 3/s \leq \delta_k < 2 - 2/s$ . By Hölder's inequality, we have

$$\begin{aligned} \|u_i \partial_{x_i} u\|_{W_{\beta,\delta}^{0,s}(\mathcal{G})} &\leq \left\| \prod_j \rho_j^{\beta_j-2+3/s} \prod_k \left(\frac{r_k}{\rho}\right)^{\sigma_k} u_i \right\|_{L^\infty(\mathcal{G})} \left\| \prod_j \rho_j^{2-3/s} \prod_k \left(\frac{r_k}{\rho}\right)^{\delta_k-\sigma_k} \partial_{x_i} u \right\|_{L^s(\mathcal{G})^3} \\ &\leq c \left\| \prod_j \rho_j^{\beta_j-2+3/s} \prod_k \left(\frac{r_k}{\rho}\right)^{\sigma_k} u_i \right\|_{L^\infty(\mathcal{G})} \|\partial_{x_i} u\|_{W_{\beta-1,\delta'-1}^{0,s}(\mathcal{G})^3}. \end{aligned}$$

For the last inequality, we used the fact that  $\beta - 1 \leq 2 - 3/s$  and  $\delta'_k - 1 \leq \delta_k - \sigma_k$ . Hence,  $(u, p)$  is a solution of the problem (3.2), (3.3), where  $\Phi = F - (u \cdot \nabla) u$  is a functional of the form (1.7) with  $f \in W_{\beta,\delta}^{0,2}(\mathcal{G})$ ,  $\phi_j \in W_{\beta,\delta}^{1-1/s,s}(\mathcal{G})^{d_j}$ . Applying Theorem 3.2, we obtain  $u \in W_{\beta,\delta}^{2,s}(\mathcal{G})^3$  and  $p \in W_{\beta,\delta}^{1,s}(\mathcal{G})$ .  $\square$

### 3.4 Regularity results in weighted Hölder spaces

In order to extend the results of Theorems 3.3 and 3.4 to problem (1.2), (1.3), we consider first the nonlinear term in the Navier-Stokes system.

**Lemma 3.3** *Let  $u \in C_{\beta,\delta}^{2,\sigma}(\mathcal{G})$ , where  $\beta_j \leq 3 + \sigma$  for  $j = 1, \dots, d$ ,  $0 \leq \delta_k < 2 + \sigma$ ,  $\delta_k - \sigma$  is not an integer for  $k = 1, \dots, m$ . Then  $u \partial_{x_i} u \in C_{\beta,\delta'}^{0,\sigma}(\mathcal{G})$  for every  $\delta'$  such that  $\delta'_k \geq \max(0, \delta_k - 1)$ ,  $k = 1, \dots, m$ .*

*P r o o f.* We have to show that there exists a constant  $C$  such that

$$\prod_j \rho_j(x)^{\beta_j-\sigma} \prod_k \left(\frac{r_k}{\rho}\right)^{\max(\delta'_k-\sigma,0)} |u(x) \partial_{x_i} u(x)| \leq C, \quad (3.7)$$

$$\prod_j \rho_j(x)^{\beta_j} \prod_k \left(\frac{r_k(x)}{\rho(x)}\right)^{\delta'_k} \frac{|u(x) \partial_{x_i} u(x) - u(y) \partial_{y_i} u(y)|}{|x-y|^\sigma} \leq C \quad \text{for } |x-y| < r(x)/2 \quad (3.8)$$

and

$$\rho_j(x)^{\beta_j-\delta'_k} \frac{|u(x) \partial_{x_i} u(x) - u(y) \partial_{y_i} u(y)|}{|x-y|^{\sigma-\delta'_k}} \leq C \quad \text{for } \delta'_k < \sigma, \quad x, y \in \mathcal{G}_{j,k}, \quad |x-y| < \rho_j(x)/2. \quad (3.9)$$

Here  $\mathcal{G}_{j,k} = \{x \in \mathcal{G} : \rho_j(x) < 3\rho(x)/2, \quad r_k(x) < 3r(x)/2\}$ . Inequality (3.7) follows immediately from the estimates

$$\prod_j \rho_j(x)^{\beta_j-2-\sigma+|\alpha|} \prod_k \left(\frac{r_k}{\rho}\right)^{\max(\delta_k-2-\sigma+|\alpha|,0)} |\partial_x^\alpha u(x)| \leq \|u\|_{C_{\beta,\delta}^{2,\sigma}(\mathcal{G})} \quad \text{for } |\alpha| \leq 2 \quad (3.10)$$

and the inequalities  $\beta_j - \sigma < 3$ ,  $\delta'_k \geq \delta_k - 1$ . Furthermore for  $|x-y| < r(x)/2$ , we have

$$\frac{|u(x) - u(y)|}{|x-y|^\sigma} \leq |x-y|^{1-\sigma} |\nabla u(x + t(y-x))| \leq c r(x)^{1-\sigma} \prod_j \rho_j(x)^{1+\sigma-\beta_j} \prod_k \left(\frac{r_k(x)}{\rho(x)}\right)^{-\max(\delta_k-1-\sigma,0)}$$

and analogously

$$\frac{|\partial_{x_i} u(x) - \partial_{y_i} u(y)|}{|x-y|^\sigma} \leq c r(x)^{1-\sigma} \prod_j \rho_j(x)^{\sigma-\beta_j} \prod_k \left(\frac{r_k(x)}{\rho(x)}\right)^{-\max(\delta_k-\sigma,0)}.$$

Hence,

$$\begin{aligned} \frac{|u(x) \partial_{x_i} u(x) - u(y) \partial_{y_i} u(y)|}{|x-y|^\sigma} &\leq \frac{|u(x) - u(y)|}{|x-y|^\sigma} |\partial_{x_i} u(x)| + \frac{|\partial_{x_i} u(x) - \partial_{y_i} u(y)|}{|x-y|^\sigma} |u(y)| \\ &\leq c r(x)^{1-\sigma} \prod_j \rho_j(x)^{2+2\sigma-2\beta_j} \left( \prod_k \left(\frac{r_k(x)}{\rho(x)}\right)^{-2\max(\delta_k-1-\sigma,0)} + \prod_k \left(\frac{r_k(x)}{\rho(x)}\right)^{-\max(\delta_k-\sigma,0)} \right) \end{aligned}$$

for  $|x - y| < r(x)/2$ . From this estimate and from the inequalities

$$c_1 r(x) \leq \prod_j \rho_j(x) \prod_k \frac{r_k(x)}{\rho(x)} \leq c_2 r(x),$$

$\delta'_k + 1 - \sigma \geq 2 \max(\delta_k - \sigma - 1, 0)$ ,  $\delta'_k + 1 - \sigma \geq \max(\delta_k - \sigma, 0)$ , and  $\beta_j - \sigma < 3$ , we obtain (3.8). Analogously, we obtain the estimate

$$\rho_j(x)^{\beta_j - \delta'_k} \frac{|u(x) - u(y)|}{|x - y|^{\sigma - \delta'_k}} |\partial_{x_i} u(x)| \leq C \quad \text{for } \delta'_k < \sigma, \ x, y \in \mathcal{G}_{j,k}, \ |x - y| < \rho_j(x)/2.$$

Since  $\partial_{x_i} u(x) \in C_{\beta,\delta}^{1,\sigma}(\mathcal{G}) \subset C_{\beta,\delta'+1}^{1,\sigma}(\mathcal{G})$ , there exists a constant  $C$  such that

$$\rho_j(x)^{\beta_j - 1 - \delta'_k} \frac{|\partial_{x_i} u(x) - \partial_{y_i} u(y)|}{|x - y|^{\sigma - \delta'_k}} \leq C \quad \text{for } \delta'_k < \sigma, \ x, y \in \mathcal{G}_{j,k}, \ |x - y| < \rho_j(x)/2.$$

This together with (3.10) implies

$$\rho_j(x)^{\beta_j - \delta'_k} \frac{|\partial_{x_i} u(x) - \partial_{y_i} u(y)|}{|x - y|^{\sigma - \delta'_k}} |u(x)| \leq C \quad \text{for } \delta'_k < \sigma, \ x, y \in \mathcal{G}_{j,k}, \ |x - y| < \rho_j(x)/2.$$

Thus, estimate (3.9) holds. The proof is complete.  $\square$

**Theorem 3.7** *Let  $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$  be a weak solution of the problem (1.2), (1.3), and let the conditions (i)–(iv) of Theorem 3.4 are satisfied. Then  $u \in C_{\beta,\delta}^{2,\sigma}(\mathcal{G})^3$  and  $p \in C_{\beta,\delta}^{1,\sigma}(\mathcal{G})$ .*

**P r o o f.** Suppose first that  $\delta_k > \sigma$  for  $k = 1, \dots, m$ . Let  $\varepsilon$  be an arbitrarily small positive number and  $s$  an arbitrary real number greater than 1. We put  $\beta'_j = \beta_j - \sigma - 3/s + \varepsilon$  for  $j = 1, \dots, N$  and  $\delta'_k = \delta_k - \sigma - 2/s + \varepsilon$ . From our assumptions on  $f, g, h_j$  and  $\phi_j$  it follows that

$$f \in W_{\beta',\delta'}^{0,s}(\mathcal{G})^3, \quad g \in W_{\beta',\delta'}^{1,s}(\mathcal{G}), \quad h_j \in W_{\beta',\delta'}^{2-1/s,s}(\Gamma_j)^{3-d_j}, \quad \phi_j \in W_{\beta',\delta'}^{1-1/s,s}(\Gamma_j)^{d_j}.$$

Using Theorem 3.6, we obtain  $u \in W_{\beta',\delta'}^{2,s}(\mathcal{G})^3$ . Consequently,

$$u \in W_{\beta'-2,-2/s+\varepsilon}^{0,s}(\mathcal{G})^3, \quad \partial_{x_i} u \in W_{\beta'-1,\delta''}^{0,s}(\mathcal{G})^3, \quad \partial_{x_i} \partial_{x_j} u \in W_{\beta',\delta'}^{0,s}(\mathcal{G})^3$$

for  $i, j = 1, 2, 3$ , where  $\delta''_k = \max(\delta_k - \sigma - 1, 0) - 2/s + \varepsilon$  for  $k = 1, \dots, m$ . This together with Hölder's inequality implies

$$u_j \partial_{x_j} u \in W_{2\beta'-3,\delta''-2/s+\varepsilon}^{0,s/2}(\mathcal{G})^3, \quad u_j \partial_{x_i} \partial_{x_j} u \in W_{2\beta'-2,\delta'-2/s+\varepsilon}^{0,s/2}(\mathcal{G})^3, \quad (\partial_{x_i} u_j) \partial_{x_j} u \in W_{2\beta'-2,2\delta''}^{0,s/2}(\mathcal{G})^3.$$

Therefore,  $u_j \partial_{x_j} u \in W_{2\beta'-2,\delta'-2/s+\varepsilon}^{1,s/2}(\mathcal{G})^3$ . The numbers  $\varepsilon$  and  $s$  can be chosen such that  $\beta_j - \sigma \leq 3 - 2\varepsilon$  for  $j = 1, \dots, d$ ,  $\delta_k - \sigma > 2/s$  for  $k = 1, \dots, m$ ,  $\varepsilon + 1/s \leq 1/2$ , and  $1 - 6/s > \sigma$ . Then  $2\beta'_j - 2 \leq \beta_j - \sigma + 1 - 6/s$ ,  $\delta'_k - 2/s + \varepsilon \leq \delta_k - \sigma + 1 - 6/s$ , and  $\delta_k - \sigma + 1 - 6/s > 1 - 4/s$ . Consequently,

$$u_j \partial_{x_j} u \in W_{\beta-\sigma+1-6/s,\delta-\sigma+1-6/s}^{1,s/2}(\mathcal{G})^3 = V_{\beta-\sigma+1-6/s,\delta-\sigma+1-6/s}^{1,s/2}(\mathcal{G})^3 \subset \mathcal{N}_{\beta,\delta}^{0,\sigma}(\mathcal{G})^3$$

(see Lemma 2.4). Hence,  $(u, p)$  is a solution of the problem (3.5), (3.6), where  $f' = f - (u \cdot \nabla) u \in C_{\beta,\delta}^{0,\sigma}(\mathcal{G})^3$ . Applying Theorem 3.4, we obtain  $(u, p) \in C_{\beta,\delta}^{2,\sigma}(\mathcal{G})^3 \times C_{\beta,\delta}^{1,\sigma}(\mathcal{G})$ .

Suppose now that  $\delta'_k < \sigma$  for at least one  $k$ . By the first part of the proof, we obtain  $u \in C_{\beta,\gamma}^{2,\sigma}(\mathcal{G})^3$ , where  $\gamma_k = \max(\delta_k, \sigma + \varepsilon)$ ,  $\varepsilon$  is an arbitrarily small positive real number. Then Lemma 3.3 implies  $f' = f - (u \cdot \nabla) u \in C_{\beta,\delta}^{0,\sigma}(\mathcal{G})^3$ , and from Theorem 3.4 it follows that  $(u, p) \in C_{\beta,\delta}^{2,\sigma}(\mathcal{G})^3 \times C_{\beta,\delta}^{1,\sigma}(\mathcal{G})$ . The proof is complete.  $\square$

Finally, we prove the analogous  $C_{\beta,\delta}^{1,\sigma}$ -regularity result.

**Theorem 3.8** *Let  $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$  be a weak solution of the problem (1.2), (1.3). Suppose that conditions (i)–(iv) of Theorem 3.3 are satisfied. Then  $u \in C_{\beta,\delta}^{1,\sigma}(\mathcal{G})^3$  and  $p \in C_{\beta,\delta}^{0,\sigma}(\mathcal{G})$ .*

**P r o o f.** 1) Let first  $\delta_k > \sigma$  for  $k = 1, \dots, m$ . Then  $g \in W_{\beta',\delta'}^{0,s}(\mathcal{G})$  and  $h_j \in W_{\beta',\delta'}^{1-1/s,s}(\Gamma_j)^{3-d_j}$ , where  $\beta'_j = \beta_j - \sigma - 3/s + \varepsilon$ ,  $\delta'_k = \delta_k - \sigma - 2/s + \varepsilon$ ,  $\varepsilon$  is an arbitrarily small positive number, and  $s > 1$ . Furthermore, the functional (1.7) belongs to  $V_{\beta',\delta'}^{-1,s}(\mathcal{G})^3$ . Using Theorem 3.5, we obtain  $(u, p) \in W_{\beta',\delta'}^{1,s}(\mathcal{G})^3 \times W_{\beta',\delta'}^{1,s}(\mathcal{G})^3$ . We consider the term

$$(u \cdot \nabla) u = \sum_{j=1}^3 \partial_{x_j} (u_j u) - \sum_{j=1}^3 u \partial_{x_j} u_j = \sum_{j=1}^3 \partial_{x_j} (u_j u) + gu.$$

From the inclusions  $u_i \in W_{2\beta'-1, -2/s+\varepsilon}^{0,s}(\mathcal{G})$ ,  $\partial_{x_k} u_i \in W_{\beta',\delta'}^{0,s}(\mathcal{G})$  it follows that

$$u_j u \in W_{2\beta'-1, \delta'-2/s+\varepsilon}^{1,s/2}(\mathcal{G})^3 \subset W_{\beta-\sigma+1-6/s, \delta-\sigma+1-6/s}^{1,s/2}(\mathcal{G})^3 = V_{\beta-\sigma+1-6/s, \delta-\sigma+1-6/s}^{1,s/2}(\mathcal{G})^3 \subset \mathcal{N}_{\beta,\delta}^{0,\sigma}(\mathcal{G})^3$$

if  $\varepsilon$  is sufficiently small and  $s$  is sufficiently large (see Lemma 2.4). Furthermore, from  $g \in \mathcal{N}_{\beta,\delta}^{0,\sigma}(\mathcal{G})$ ,

$$u \in W_{\beta', 1-2/s+\varepsilon}^{1,s}(\mathcal{G})^3 = V_{\beta', 1-2/s+\varepsilon}^{1,s}(\mathcal{G})^3 \subset \mathcal{N}_{\beta-1+\varepsilon, \sigma+\varepsilon+1/s}^{0,\sigma}(\mathcal{G})^3$$

and Lemma 2.7 it follows that

$$gu \in \mathcal{N}_{2\beta-1+\varepsilon-\sigma, \delta+\varepsilon+1/s}^{0,\sigma}(\mathcal{G}) \subset \mathcal{N}_{\beta+1, \delta+1}^{0,\sigma}(\mathcal{G}).$$

Consequently, we have  $(u \cdot \nabla) u \in C_{\beta,\delta}^{-1,\sigma}(\mathcal{G})^3$ , and Theorem 3.3 implies  $(u, p) \in C_{\beta,\delta}^{1,\sigma}(\mathcal{G})^3 \times C_{\beta,\delta}^{0,\sigma}(\mathcal{G})$ .

2) Suppose that  $\delta_k < \sigma$  for some  $k$ . By the first part of the proof, we have  $(u, p) \in C_{\beta,\gamma}^{1,\sigma}(\mathcal{G})^3 \times C_{\beta,\gamma}^{0,\sigma}(\mathcal{G})$ , where  $\gamma_k = \max(\delta_k, \sigma + \varepsilon)$  for  $k = 1, \dots, m$ ,  $\varepsilon$  is an arbitrarily small positive number. In particular,  $u_j \in \mathcal{N}_{\beta-1,\gamma}^{0,\sigma}(\mathcal{G})$ ,  $\partial_{x_j} u \in \mathcal{N}_{\beta,\gamma}^{0,\sigma}(\mathcal{G})^3$ , and therefore (by Lemma 2.7)

$$u_j \partial_{x_j} u \in \mathcal{N}_{2\beta-1-\sigma, 2\gamma-\sigma}^{0,\sigma}(\mathcal{G})^3.$$

The last space is contained in  $\mathcal{N}_{\beta+1, \delta+1}^{0,\sigma}(\mathcal{G})^3$  for sufficiently small  $\varepsilon$ . Applying Theorem 3.3, we obtain  $(u, p) \in C_{\beta,\delta}^{1,\sigma}(\mathcal{G})^3 \times C_{\beta,\delta}^{0,\sigma}(\mathcal{G})$ .  $\square$

### 3.5 Necessity of the conditions on $\beta$ and $\delta$

Let  $\Lambda_j$  be the eigenvalue of the pencil  $\mathfrak{A}_j(\lambda)$  with smallest real part  $> -1/2$ . We show that the inequalities

$$\beta_j + 3/s > 2 - \operatorname{Re} \Lambda_j, \quad \delta_k + 2/s > 2 - \mu_k \quad (3.11)$$

in Theorem 3.6 cannot be weakened.

We assume first that  $\beta_j + 3/s \leq 2 - \operatorname{Re} \Lambda_j$  for some  $j$  and that  $\delta$  satisfies the second condition of (3.11). By our assumptions on the domain, there exist a neighborhood  $\mathcal{U}_j$  of  $x^{(j)}$  and a diffeomorphism  $\kappa$  mapping  $\mathcal{G} \cap \mathcal{U}_j$  onto the intersection of a cone  $\mathcal{K}_j$  with the unit ball such that  $\kappa'(x^{(j)}) = I$ . In the new coordinates  $y = \kappa(x)$ , the Navier-Stokes system takes the form

$$\sum_{i,j=1}^3 a_{i,j}(y) \frac{\partial^2 u}{\partial y_i \partial y_j} + \sum_{i=1}^3 a_i(y) \frac{\partial u}{\partial y_i} + (u \cdot \kappa' \nabla_y) u + \kappa' \nabla_y p = f, \quad \kappa' \nabla_y \cdot u = g,$$

where  $a_{i,j}(0) = -\nu \delta_{i,j}$ . Here by  $\kappa'$  we mean the matrix  $\kappa'(\kappa^{-1}(y))$ . We consider the functions

$$u = \zeta(y) |y|^{\Lambda_j} \Phi(y/|y|), \quad p = \zeta(y) |y|^{\Lambda_j-1} \Psi(y/|y|),$$

where  $(\Phi, \Psi)$  is an eigenvector of the pencil  $\mathfrak{A}_j(\lambda)$  corresponding to the eigenvalue  $\Lambda_j$  and  $\zeta$  is a smooth cut-off function equal to one near the origin. The eigenvector  $(\Phi, \Psi)$  belongs to the space  $W_{\gamma}^{2,t}(\Omega_j)^3 \times W_{\gamma}^{1,t}(\Omega_j)$

with arbitrary  $t$  and  $\gamma$  satisfying  $\max(2 - \mu_k, 0) < \gamma_k + 2/t < 2$ . Here,  $W_\gamma^{l,t}(\Omega_j)$  is the closure of the set  $C^\infty(\bar{\Omega}_j)$  with respect to the norm

$$\|u\|_{W_\gamma^{2,t}(\Omega_j)} = \left( \int_{\mathcal{K}_j} \sum_{|\alpha| \leq l} \prod_k r_k^{t\delta_k} |\partial_x^\alpha u(x)|^t dx \right)^{1/t},$$

$1/2 < |x| < 2$

where  $u$  is extended by  $u(x) = u(x/|x|)$  to  $\mathcal{K}_j$ . In particular,  $\Phi \in L_\infty(\Omega_j)^3$  and  $\Phi_k \partial_{x_j} \Phi \in W_\delta^{0,s}(\Omega_j)^3$  for  $k = 1, 2, 3$ . Since  $(\Phi, \Psi)$  is an eigenvector, the vector function  $|y|^{\Lambda_j} (\Phi(y/|y|), |y|^{-1} \Psi(y/|y|))$  is a solution of the linear Stokes system with zero right-hand sides. From this and from the equalities  $\kappa'(x^{(j)}) = I$  and  $a_{i,j}(0) = -\nu \delta_{i,j}$ , it follows that  $f \in W_{\beta_j, \delta}^{0,t}(\mathcal{K}_j)^3$  and  $g \in W_{\beta_j, \delta}^{1,t}(\mathcal{K}_j)$  if  $\beta_j + 3/s > 1 - \operatorname{Re} \Lambda_j$ ,  $\beta_j + 3/s > 1 - 2 \operatorname{Re} \Lambda_j$ . Analogously, the corresponding boundary data are from  $W_{\beta_j, \delta}^{1-1/t, t}$  and  $W_{\beta_j, \delta}^{1-1/t, t}$ , respectively. However,  $u \notin W_{\beta_j, \delta}^{2,t}(\mathcal{K}_j)^3$  for  $\beta_j + 3/s \leq 2 - \operatorname{Re} \Lambda_j$ . This example shows that the inequality  $\beta_j + 3/s > 2 - \operatorname{Re} \Lambda_j$  cannot be weakened.

Now we show that the inequality  $\delta_k + 2/s > 2 - \mu_k$  cannot be weakened. We assume for the sake of simplicity that the edge  $M_k$  is a part of the  $x_3$ -axis and that the adjacent faces  $\Gamma_{k+}$  and  $\Gamma_{k-}$  are plane. Let  $\delta_k + 2/s \leq 2 - \mu_k$ , and let  $\lambda_k$  be an eigenvalue of the pencil  $A_k(\lambda)$  with the real part  $\mu_k$ . Then we consider the functions

$$u(x) = \zeta(x) r^{\lambda_k} \Phi(\varphi), \quad p(x) = \zeta(x) r^{\lambda_k - 1} \Psi(\varphi),$$

where  $r, \phi$  are the polar coordinates in the  $(x_1, x_2)$ -plane,  $(\Phi, \Psi)$  is an eigenvector of the pencil  $A_k(\lambda)$  corresponding to the eigenvalue  $\lambda_k$ , and  $\zeta$  is a smooth cut-off function equal to one in a neighborhood of a certain point  $x^{(0)} \in M_k$  and equal to zero near the other edges. Since the vector function  $r^{\Lambda_k} (\Phi(\varphi), r^{-1} \Psi(\varphi))$  is a solution of the linear Stokes system with zero right-hand sides, it follows that

$$-\nu \Delta u + (u \cdot \nabla) u + \nabla p \in W_{\beta, \delta}^{0,s}(\mathcal{G})^3, \quad \nabla \cdot u \in W_{\beta, \delta}^{1,s}(\mathcal{G})$$

if  $\delta_k + 2/s > 1 - \mu_k$ . However,  $u \notin W_{\beta, \delta}^{2,s}(\mathcal{G})^3$  for  $\delta_k + 2/s \leq 2 - \mu_k$ . This means, the result of Theorem 3.6 fails if  $1 - \mu_k < \delta_k + 2/s \leq 2 - \mu_k$ .

Analogously, it can be shown that the inequalities for  $\beta_j$  and  $\delta_k$  in Theorems 3.5, 3.7 and 3.8 cannot be weakened.

### 3.6 Examples

Here, we establish some regularity results for weak solutions in the class of the nonweighted spaces  $W^{l,s}(\mathcal{G})$  and  $C^{l,\sigma}(\mathcal{G})$ . We assume that  $\mathcal{G}$  is a polyhedron with faces  $\Gamma_j$ ,  $j = 1, \dots, N$ , and edges  $M_k$ ,  $k = 1, \dots, m$ . The angle at the edge  $M_k$  is denoted by  $\theta_k$ . For the sake of simplicity, we restrict ourselves to the case  $g = 0$  and to homogeneous boundary conditions

$$S_j u = 0, \quad N_j(u, p) = 0 \quad \text{on } \Gamma_j, \quad j = 1, \dots, N. \quad (3.12)$$

Analogous results are valid for inhomogeneous boundary conditions provided the boundary data satisfy certain compatibility conditions on the edges. Note that there are the following equalities

$$W^{1,s}(\mathcal{G}) = V_{0,0}^{1,s}(\mathcal{G}) \quad \text{if } s < 2, \quad W^{1,s}(\mathcal{G}) = W_{0,0}^{1,s}(\mathcal{G}) \quad \text{if } s < 3.$$

*The Dirichlet problem.* For arbitrary  $f \in W^{-1,2}(\mathcal{G})^3$ , there exists a solution  $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$  of the Dirichlet problem

$$-\nu \Delta u + (u \cdot \nabla) u + \nabla p = f, \quad -\nabla u = 0 \text{ in } \mathcal{G}, \quad u = 0 \text{ on } \Gamma_j, \quad j = 1, \dots, N.$$

(see e.g. [5, Th.IV.2.1]). This solution is unique for sufficiently small  $f$ .

The regularity results established below are based on the following properties of the operator pencils  $\mathfrak{A}_j(\lambda)$  (see [11] or [10, Ch.5]).

- The strip  $-1/2 \leq \operatorname{Re} \lambda \leq 0$  is free of eigenvalues of the pencils  $\mathfrak{A}_j(\lambda)$ .

- If the cone  $\mathcal{K}_j$  is contained in a half-space, then the strip  $-1/2 \leq \operatorname{Re} \lambda \leq 1$  contains only the eigenvalue  $\lambda = 1$  of the pencil  $\mathfrak{A}_j(\lambda)$ . This eigenvalue has only the eigenvector  $(0, 0, 0, c)$ ,  $c = \text{const.}$ , and no generalized eigenvectors.
- The eigenvalues of the pencil  $\mathfrak{A}_j(\lambda)$  in the strip  $-1/2 \leq \operatorname{Re} \lambda \leq 1$  are real and monotonous with respect to the cone  $\mathcal{K}_j$ .

Moreover, the eigenvalues for a circular cone are solutions of a certain transcendental equation (see [11] or [10, Section 5.6]).

The numbers  $\mu_k$  can be easily calculated. In the case  $\theta_k < \pi$ , we have  $\mu_k = \pi/\theta_k$ . If  $\theta_k > \pi$ , then  $\mu_k$  is the smallest positive solution of the equation

$$\sin(\mu\theta_k) + \mu \sin \theta_k = 0. \quad (3.13)$$

Note that  $\mu_k > 1/2$  for every  $\theta_k < 2\pi$ ,  $\mu_k > 2/3$  if  $\theta_k < 3 \arccos \frac{1}{4} \approx 1.2587\pi$ ,  $\mu_k > 1$  if  $\theta_k < \pi$ , and  $\mu_k > 4/3$  if  $\theta_k < \frac{3}{4}\pi$ . Using these facts together with Theorems 3.5 and 3.6, we obtain the following assertions.

- If  $f \in (W^{1,s'}(\mathcal{G})^*)^3$ ,  $2 < s \leq 3$ ,  $s' = s/(s-1)$ , then  $(u, p) \in W^{1,s}(\mathcal{G})^3 \times L_s(\mathcal{G})$ . If the polyhedron  $\mathcal{G}$  is convex, then this assertion is true for all  $s > 2$ .
- If  $f \in W^{-1,2}(\mathcal{G})^3 \cap L_s(\mathcal{G})^3$ ,  $1 < s \leq 4/3$ , then  $(u, p) \in W^{2,s}(\mathcal{G})^3 \times W^{1,s}(\mathcal{G})$ . If  $\theta_k < 3 \arccos \frac{1}{4} \approx 1.2587\pi$  for  $k = 1, \dots, m$ , then this result is true for  $1 < s \leq 3/2$ . If  $\mathcal{G}$  is convex, then this result is valid for  $1 < s \leq 2$ . If, moreover, the angles at the edges are less than  $\frac{3}{4}\pi$ , then the result holds even for  $1 < s < 3$ .

Furthermore, the following assertion is valid.

- If  $\mathcal{G}$  is convex,  $f \in C^{-1,\sigma}(\mathcal{G})$ , and  $\sigma$  is sufficiently small (such that  $1 + \sigma < \pi/\theta_k$  and there are no eigenvalues of the pencils  $\mathfrak{A}_j$  in the strip  $1 < \operatorname{Re} \lambda \leq 1 + \sigma$ ), then  $(u, p) \in C^{1,\sigma}(\mathcal{G})^3 \times C^{0,\sigma}(\mathcal{G})$ .

We prove the last result. Let  $\varepsilon$  be a positive number,  $\varepsilon < 1 - \sigma$ . Since  $C^{-1,\sigma}(\mathcal{G}) \subset C_{\sigma+\varepsilon,0}^{-1,\sigma}(\mathcal{G})$ , it follows from Theorem 3.8 that  $(u, p) \in C_{\sigma+\varepsilon,0}^{1,\sigma}(\mathcal{G}) \times C_{\sigma+\varepsilon,0}^{0,\sigma}(\mathcal{G})$  if  $\sigma < \mu_k - 1$ . In particular, we have  $u \in C_{\sigma+1,0}^{1,\sigma}(\mathcal{G}) \subset C^{0,\sigma}(\mathcal{G})$ . This implies  $u_j u \in C^{0,\sigma}(\mathcal{G})^3$ . Since  $\nabla \cdot u = 0$ , it follows that  $(u \cdot \nabla)u = \sum_j \partial_{x_j}(u_j u) \in C^{-1,\sigma}(\mathcal{G})^3$ . Thus,  $(u, p)$  satisfies the Stokes system

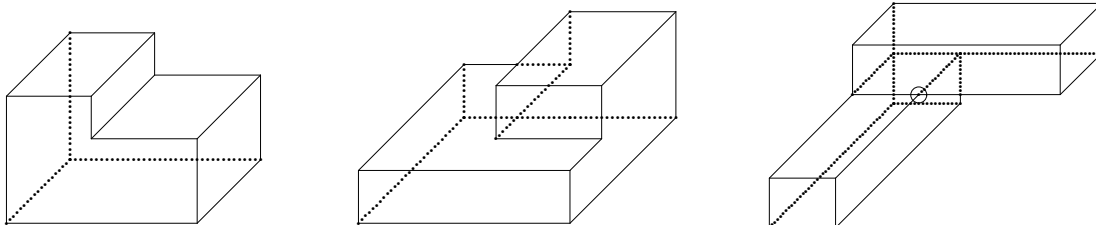
$$-\nu \Delta u + \nabla p = f', \quad -\nabla \cdot u = 0,$$

where  $f' = f - (u \cdot \nabla)u \in C^{-1,\sigma}(\mathcal{G}) \subset C_{0,0}^{-1,\sigma}(\mathcal{G})$ . Hence by [19, Th.4.3], the solution  $(u, p)$  admits the decomposition

$$\begin{pmatrix} u(x) \\ p(x) \end{pmatrix} = \begin{pmatrix} 0 \\ p(x^{(k)}) \end{pmatrix} + \begin{pmatrix} w(x) \\ q(x) \end{pmatrix}$$

in a neighborhood of the vertex  $x^{(k)}$ , where  $(w, q) \in C_{0,0}^{1,\sigma}(\mathcal{G})^3 \times C_{0,0}^{0,\sigma}(\mathcal{G})$ . This is true for every vertex  $x^{(k)}$ ,  $k = 1, \dots, d$ . Consequently,  $(u, p) \in C^{1,\sigma}(\mathcal{G})^3 \times C^{0,\sigma}(\mathcal{G})$ .

For special domains, it is possible to obtain precise regularity results. Let for example  $\mathcal{G}$  have the form of steps as in the first two pictures below with angles  $\pi/2$  or  $3\pi/2$  at every edge or the form of two beams, where one lies on the other as in the third picture. Note that the third polyhedron is not Lipschitz. The greatest edge angle is  $3\pi/2$ , and we obtain  $\min \mu_k = 0.54448373\dots$ . Moreover for every vertex, there



exists a circular cone with the same vertex and aperture  $3\pi/2$  which contains the polyhedron. The left polyhedron is even contained in a half-space bounded by a plane through an arbitrary of the vertices. Consequently, the smallest positive eigenvalue of the pencils  $\mathfrak{A}_j(\lambda)$  does not exceed the first eigenvalue for a circular cone with vertex  $3\pi/2$ . A numerical calculation shows that this eigenvalue is greater than  $(3 \min \mu_k - 1)/2$ . This means that for  $\beta = 0$  and  $\delta = 0$ , the condition (iii) in Theorems 3.5 and 3.6 is stronger than the condition (ii) in the same theorems. Thus, we obtain

$$\begin{aligned} (u, p) &\in W^{1,s}(\mathcal{G})^3 \times L_s(\mathcal{G}) && \text{if } f \in W^{-1,s}(\mathcal{G}), s < 2/(1 - \min \mu_k) = 4.3905\dots, \\ (u, p) &\in W^{2,s}(\mathcal{G})^3 \times W^{1,s}(\mathcal{G}) && \text{if } f \in L_s(\mathcal{G}), s < 2/(2 - \min \mu_k) = 1.3740\dots \end{aligned}$$

Here the condition on  $s$  is sharp.

We give some comments concerning the examples in the introduction (the flow outside a regular polyhedron  $G$ ). By Theorem 3.6, the regularity result  $(u, p) \in W^{2,s} \times W^{1,s}$  in an arbitrary bounded subdomain of the complement of  $G$  holds if  $s < 2/(2 - \mu_k)$  and there are no eigenvalues of the pencils  $\mathfrak{A}_j(\lambda)$  in the strip  $-1/2 < \operatorname{Re} \lambda < 2 - 3/s$ . Here,  $\mu_k$  is the smallest positive solution of the equation (3.13), where  $\theta_k = \theta$  is the edge angle in the exterior of  $G$ ,  $\sin \theta$  is equal to  $-\frac{2}{3}\sqrt{2}$  if  $G$  is a regular tetrahedron or octahedron,  $-1$  if  $G$  is a cube,  $-\frac{2}{5}\sqrt{5}$  if  $G$  is a regular dodecahedron, and  $-2/3$  if  $G$  is a regular icosahedron. The smallest positive solutions of (3.13) are  $\mu_k = 0.52033360\dots$  for the regular tetrahedron,  $\mu_k = 0.54448373\dots$  for a cube,  $\mu_k = 0.58489758\dots$  for the regular octahedron,  $\mu_k = 0.60487306\dots$  for the regular dodecahedron, and  $\mu_k = 0.68835272\dots$  for the regular icosahedron. In the case of a regular tetrahedron, cube, octahedron or dodecahedron, the inequality  $s < 2/(2 - \mu_k)$  implies  $2 - 3/s < 3\mu_k/2 - 1 < 0$ . Then the absence of eigenvalues of the pencil  $\mathfrak{A}_j(\lambda)$  in the strip  $-1/2 < \operatorname{Re} \lambda < 2 - 3/s$  follows from [10, Th.5.5.6]. The exterior of a regular icosahedron is contained in a right circular cone with aperture less than  $255^\circ$ . Numerical results for right circular cones together with the monotonicity of the eigenvalues of the pencils  $\mathfrak{A}_j(\lambda)$  in the interval  $[-1/2, 1]$  show that also for this polyhedron, the strip  $-1/2 < \operatorname{Re} \lambda < 3\mu_k/2 - 1$  is free of eigenvalues of the pencil  $\mathfrak{A}_j(\lambda)$ . Thus, the above mentioned regularity result holds for all  $s < 2/(2 - \mu_k)$ . This inequality cannot be weakened.

*The Neumann problem for the Navier-Stokes system.* We consider a weak solution  $u \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$  of the Neumann problem

$$-\Delta u + (u \cdot \nabla) u + \nabla p = f, \quad -\nabla u = 0 \text{ in } \mathcal{G}, \quad -pn + 2\nu \varepsilon_n(u) = 0 \text{ on } \Gamma_j, \quad j = 1, \dots, N.$$

For this problem it is known that the strip  $-1 \leq \operatorname{Re} \lambda \leq 0$  contains only the eigenvalues  $\lambda = 0$  and  $\lambda = 1$  of the operator pencils  $\mathfrak{A}_j(\lambda)$  (see [10, Th.6.3.2]) if  $\mathcal{G}$  is a Lipschitz polyhedron. The numbers  $\mu_k$  are the same as for the Dirichlet problem. Therefore, the following assertions are valid.

- If  $f \in (W^{1,s'}(\mathcal{G})^*)^3$ ,  $s' = s/(s-1)$ ,  $2 < s < 3$ , then  $(u, p) \in W^{1,s}(\mathcal{G})^3 \times L_s(\mathcal{G})$ .
- If  $f \in (W^{1,2}(\mathcal{G})^*)^3 \cap L_s(\mathcal{G})^3$ ,  $1 < s \leq 4/3$ , then  $(u, p) \in W^{2,s}(\mathcal{G})^3 \times W^{1,s}(\mathcal{G})$ . If the angles  $\theta_k$  are less than  $3 \arccos \frac{1}{4}$ , then this result is true for  $1 < s < 3/2$ .

*The mixed problem with Dirichlet and Neumann boundary conditions.* We assume that on each face  $\Gamma_j$  either the Dirichlet condition  $u = 0$  or the Neumann condition  $\frac{\partial u}{\partial n} = 0$  is given. If on both adjoining faces of the edge  $M_k$  the same boundary conditions are given, then  $\mu_k > 1/2$ . If on one of the adjoining faces the Dirichlet condition and on the other face the Neumann condition is given, then  $\mu_k > 1/4$ . This implies the following result.

- If  $f \in (W^{1,2}(\mathcal{G})^*)^3 \cap L_s(\mathcal{G})^3$ ,  $1 < s \leq 8/7$ , then every weak solution  $(u, p)$  belongs to  $W^{2,s}(\mathcal{G})^3 \times W^{1,s}(\mathcal{G})$ .

*The mixed problem with boundary conditions (i)–(iii).* Let  $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$  be a weak solution of the problem (1.2), (1.3), where  $g = 0$ ,  $h_j = 0$ ,  $\phi_j = 0$  for  $j = 1, \dots, N$ , and  $d_k \leq 2$  for all  $k$  (i.e., the Neumann condition does not appear in the boundary conditions). We assume that the Dirichlet condition is given on at least one of the adjoining faces of every edge. Then, by [10, Th.6.1.5], the strip  $-1 \leq \operatorname{Re} \lambda \leq 0$  is free of eigenvalues of the pencils  $\mathfrak{A}_j(\lambda)$ . Furthermore, we have  $\mu_k > 1/2$  if the Dirichlet condition is given on both adjoining faces of the edge  $M_k$ . For the other indices  $k$ , we have  $\mu_k > 1/4$  and  $\mu_k > 1/3$  if  $\theta_k < \frac{3}{2}\pi$ .

- If  $f \in (W^{1,s'}(\mathcal{G})^*)^3$ ,  $2 < s \leq 8/3$ , then  $(u, p) \in W^{1,s}(\mathcal{G})^3 \times L_s(\mathcal{G})$ . Suppose that  $\theta_k < \frac{3}{2}\pi$  if the boundary condition (ii) or (iii) is given on one of the adjoining faces of the edge  $M_k$ . Then this result is even true for  $2 < s \leq 3$ .
- If  $f \in (W^{1,2}(\mathcal{G})^*)^3 \cap L_s(\mathcal{G})^3$ ,  $1 < s \leq 8/7$ , then  $(u, p) \in W^{2,s}(\mathcal{G})^3 \times W^{1,s}(\mathcal{G})$ . Suppose that  $\theta_k < 3 \arccos \frac{1}{4}$  if the Dirichlet condition is given on both adjoining faces of  $M_k$ ,  $\theta_k < \frac{3}{2} \arccos \frac{1}{4}$  if the boundary condition (ii) is given on one of the adjoining faces of  $M_k$ , and  $\theta_k < \frac{3}{4}\pi$  if the boundary condition (iii) is given on one of the adjoining faces of  $M_k$ . Then the last result is true for  $1 < s \leq 3/2$ .

Note that in the last case, we have  $\mu_k > 2/3$  for  $k = 1, \dots, m$ .

Finally, we assume that the homogeneous Dirichlet condition  $u = 0$  is given on the faces  $\Gamma_1, \dots, \Gamma_{N-1}$ , while the homogeneous boundary condition (iii) is given on  $\Gamma_N$ . Let  $I$  be the set of all  $k$  such that  $M_k \subset \bar{\Gamma}_N$  and  $I' = \{1, \dots, m\} \setminus I$ . We suppose that the polyhedron  $\mathcal{G}$  is convex and  $\theta_k < \pi/2$  for  $k \in I$ . Then  $\mu_k > 1$  for all  $k$ , and the strip  $-1/2 \leq \operatorname{Re} \lambda \leq 1$  contains only the simple eigenvalue  $\lambda = 1$  of the pencils  $\mathfrak{A}_j(\lambda)$  (see [10, Th.6.2.7]). If  $\theta_k < \frac{3}{8}\pi$  for  $k \in I$  and  $\theta_k < \frac{3}{4}\pi$  for  $k \in I'$ , then even  $\mu_k > 4/3$ . This implies the following result.

- Let  $f \in (W^{1,2}(\mathcal{G})^*)^3 \cap L_s(\mathcal{G})^3$ ,  $1 < s \leq 2$ . Then any weak solution belongs to  $W^{2,s}(\mathcal{G})^3 \times W^{1,s}(\mathcal{G})$ . If  $\theta_k < \frac{3}{8}\pi$  for  $k \in I$  and  $\theta_k < \frac{3}{4}\pi$  for  $k \in I'$ , then the result holds even for  $1 < s < 3$ .

Furthermore analogously to the Dirichlet problem, the following assertion holds.

- Let  $f \in (W^{1,2}(\mathcal{G})^*)^3 \cap C^{-1,\sigma}(\mathcal{G})$ . Then for sufficiently small  $\sigma$ , we have  $(u, p) \in C^{1,\sigma}(\mathcal{G}) \times C^{0,\sigma}(\mathcal{G})$ .

## 4 Existence of weak solutions in $W^{1,s}(\mathcal{G}) \times L_s(\mathcal{G})$ , $s < 2$

In Section 1 we proved the existence of weak solutions of the boundary value problem in  $W^{1,2}(\mathcal{G}) \times L_2(\mathcal{G})$ . Using the regularity result of Theorem 3.5, we obtain also the existence of weak solutions in  $W^{1,s}(\mathcal{G}) \times L_s(\mathcal{G})$  for sufficiently small  $s > 2$ . In this section, we will prove that weak solutions exist also in the space  $W^{1,s}(\mathcal{G}) \times L_s(\mathcal{G})$  with  $s < 2$  provided the norms of the right-hand sides of (1.4), (1.5) in the corresponding Sobolev spaces are sufficiently small. Throughout this section, we suppose that the Dirichlet condition is given on at least one of the adjoining faces of every edge  $M_k$ .

### 4.1 Solvability of the linearized problem in a cone

Let  $\mathcal{K}$  be the same polyhedral cone as in Section 2.1. We consider weak solutions  $(u, p) \in V_{\beta,\delta}^{1,s}(\mathcal{G})^3 \times V_{\beta,\delta}^{0,s}(\mathcal{G})$  of the linear Stokes system in  $\mathcal{K}$  with boundary conditions (i)–(iv) on the faces  $\Gamma_j$ . This means that  $(u, p)$  satisfies the equations

$$b_{\mathcal{K}}(u, v) - \int_{\mathcal{K}} p \nabla \cdot v \, dx = F(v) \quad \text{for all } v \in V_{-\beta,-\delta}^{1,s'}(\mathcal{K})^3, \quad S_j v|_{\Gamma_j} = 0, \quad (4.1)$$

$$-\nabla \cdot u = g \quad \text{in } \mathcal{K}, \quad S_j u = h_j \quad \text{on } \Gamma_j, \quad j = 1, \dots, N. \quad (4.2)$$

Here  $b_{\mathcal{K}}$  denotes the bilinear form (1.6), where  $\mathcal{G}$  has to be replaced by  $\mathcal{K}$ . We define the space  $V_{\beta,\delta}^{-1,s}(\mathcal{K}; S)$  as the set of all linear and continuous functionals on the space  $\{v \in V_{-\beta,-\delta}^{1,s'}(\mathcal{K})^3 : S_j v|_{\Gamma_j} = 0\}$ , where  $s' = s/(s-1)$ . Furthermore, the pencils  $A_k(\lambda)$  for the edges  $M_k$  and  $\mathfrak{A}(\lambda)$  for the vertex of the cone  $\mathcal{K}$  are defined as in Section 3.1. If the Dirichlet condition (i) is given on at least one of the adjoining faces of the edge  $M_k$ , then  $\lambda = 0$  is not an eigenvalue of the pencil  $A_k(\lambda)$ .

The following lemma is proved in [18] under the condition  $\max(1 - \operatorname{Re} \lambda_1^{(k)}, 0) < \delta_k + 2/s < 1$ . Using the sharper estimates of Green's matrix given in [17] for the case when  $\lambda = 0$  is not an eigenvalue of the pencils  $A_k(\lambda)$ , this theorem can be proved in the same way if

$$1 - \operatorname{Re} \lambda_1^{(k)} < \delta_k + 2/s < 1 + \operatorname{Re} \lambda_1^{(k)} \quad (4.3)$$

for  $k = 1, \dots, N$ .



**Lemma 4.1** Suppose that  $F \in V_{\beta,\delta}^{-1,s}(\mathcal{K}; S)$ ,  $g \in V_{\beta,\delta}^{0,s}(\mathcal{K})$ ,  $h_j \in V_{\beta,\delta}^{1-1/s,s}(\Gamma_j)$ , there are no eigenvalues of the pencil  $\mathfrak{A}(\lambda)$  on the line  $\operatorname{Re} \lambda = 1 - \beta_j - 3/s$ , and the components of  $\delta$  satisfy the inequalities (4.3). Then there exists a unique solution  $(u, p) \in V_{\beta,\delta}^{1,s}(\mathcal{K})^3 \times V_{\beta,\delta}^{0,s}(\mathcal{K})$  of problem (4.1), (4.2).

Moreover, the following regularity results hold analogously to [18, Le.4.4 and Th.4.4].

**Lemma 4.2** 1) Suppose that in addition to the assumptions of Lemma 4.1, we have  $F \in V_{\beta',\delta'}^{-1,t}(\mathcal{K}; S)$ ,  $g \in V_{\beta',\delta'}^{0,t}(\mathcal{K})$ ,  $h_j \in V_{\beta',\delta'}^{1-1/t,t}(\Gamma_j)$ , where  $1 - \operatorname{Re} \lambda_1^{(k)} < \delta'_k + 2/t < 1 + \operatorname{Re} \lambda_1^{(k)}$  for  $k = 1, \dots, N$  and  $\beta'$  is such that there are no eigenvalues of the pencil  $\mathfrak{A}(\lambda)$  in the closed strip between the lines  $\operatorname{Re} \lambda = 1 - \beta - 3/s$  and  $\operatorname{Re} \lambda = 1 - \beta' - 3/t$ . Then the solution  $(u, p) \in V_{\beta,\delta}^{1,s}(\mathcal{K})^3 \times V_{\beta,\delta}^{0,s}(\mathcal{K})$  belongs to  $V_{\beta',\delta'}^{1,t}(\mathcal{K})^3 \times V_{\beta',\delta'}^{0,t}(\mathcal{K})$ .

2) Suppose that in addition to the assumptions of Lemma 4.1,  $g \in V_{\beta',\delta'}^{1,t}(\mathcal{K})$ ,  $h_j \in V_{\beta',\delta'}^{2-1/t,t}(\Gamma_j)$ , and the functional  $F$  has the form

$$F(v) = \int_{\mathcal{K}} f \cdot v \, dx + \sum_{j=1}^N \Phi_j \cdot v \, dx$$

with vector function  $f \in V_{\beta',\delta'}^{0,t}(\mathcal{K})$ ,  $\Phi_j \in V_{\beta',\delta'}^{1-1/t,t}(\Gamma_j)$ , where  $\beta'$  is such that there are no eigenvalues of the pencil  $\mathfrak{A}(\lambda)$  in the closed strip between the lines  $\operatorname{Re} \lambda = 1 - \beta - 3/s$  and  $\operatorname{Re} \lambda = 2 - \beta' - 3/t$ , and the components of  $\delta'$  satisfy the inequalities  $2 - \operatorname{Re} \lambda_1^{(k)} < \delta'_k + 2/t < 2 + \operatorname{Re} \lambda_1^{(k)}$ . Then the solution  $(u, p) \in V_{\beta,\delta}^{1,s}(\mathcal{K})^3 \times V_{\beta,\delta}^{0,s}(\mathcal{K})$  belongs to  $V_{\beta',\delta'}^{2,t}(\mathcal{K})^3 \times V_{\beta',\delta'}^{1,t}(\mathcal{K})$ .

## 4.2 Solvability of the linearized problem in $\mathcal{G}$

We consider the operator

$$V_{\beta,\delta}^{1,s}(\mathcal{G})^3 \times V_{\beta,\delta}^{0,s}(\mathcal{G}) \ni (u, p) \rightarrow (F, g, h) \in V_{\beta,\delta}^{-1,s}(\mathcal{G}; S) \times V_{\beta,\delta}^{0,s}(\mathcal{G}) \times \prod_j V_{\beta,\delta}^{1-1/s,s}(\Gamma_j) \quad (4.4)$$

of problem (1.9), (1.10) and denote this operator by  $\mathcal{A}_{s,\beta,\delta}$ . Here again  $V_{\beta,\delta}^{-1,s}(\mathcal{G}; S)$  is defined as the dual space of  $\{v \in V_{-\beta,-\delta}^{1,s'}(\mathcal{G})^3 : S_j v|_{\Gamma_j} = 0\}$ . We show that this operator is Fredholm if there are no eigenvalues of the pencil  $\mathfrak{A}_j(\lambda)$  on the line  $\operatorname{Re} \lambda = 1 - \beta_j - 3/s$ ,  $j = 1, \dots, d$ , and the components of  $\delta$  satisfy the inequalities

$$1 - \inf_{\xi \in M_k} \operatorname{Re} \lambda_1(\xi) < \delta_k + 2/s < 1 + \inf_{\xi \in M_k} \operatorname{Re} \lambda_1(\xi) \quad (4.5)$$

for  $k = 1, \dots, m$ . For this end, we construct a left and right regularizer for the operator  $\mathcal{A}_{s,\beta,\delta}$ .

**Lemma 4.3** Let  $\mathcal{U}$  be a sufficiently small open subset of  $\mathcal{G}$  and let  $\varphi$  be a smooth function with support in  $\bar{\mathcal{U}}$ . Suppose that there are no eigenvalues of the pencil  $\mathfrak{A}_j(\lambda)$  on the line  $\operatorname{Re} \lambda = 1 - \beta_j - 3/s$ ,  $j = 1, \dots, d$ , and the components of  $\delta$  satisfy (4.5). Then there exists an operator  $\mathcal{R}$  continuously mapping the space of all  $(F, g, h) \in V_{\beta,\delta}^{-1,s}(\mathcal{G}; S) \times V_{\beta,\delta}^{0,s}(\mathcal{K}) \times \prod_j V_{\beta,\delta}^{1-1/s,s}(\Gamma_j)$  with support in  $\bar{\mathcal{U}}$  onto  $V_{\beta,\delta}^{1,s}(\mathcal{G})^3 \times V_{\beta,\delta}^{0,s}(\mathcal{G})$  such that  $\varphi \mathcal{A}_{s,\beta,\delta} \mathcal{R}(F, g, h) = \varphi(F, g, h)$  for all  $(F, g, h)$  with support in  $\bar{\mathcal{U}}$  and  $\mathcal{R} \mathcal{A}_{s,\beta,\delta}(u, p) = (u, p)$  for all  $(u, p)$  with support in  $\bar{\mathcal{U}}$ .

**P r o o f.** Suppose first that  $\bar{\mathcal{U}}$  contains the vertex  $x^{(1)}$  of  $\mathcal{G}$ . Then there exists a diffeomorphism  $\kappa$  mapping  $\mathcal{U}$  onto a subset  $\mathcal{V}$  of a polyhedral cone  $\mathcal{K}$  with vertex at the origin such that  $\kappa(x^{(1)}) = 0$  and the Jacobian matrix  $\kappa'$  coincides with the identity matrix  $I$  at  $x^{(1)}$ . We assume that the supports of  $u$  and  $p$  are contained in  $\bar{\mathcal{U}}$ . Then the coordinate change  $y = \kappa(x)$  transforms (1.9), (1.10) into

$$\tilde{b}(\tilde{u}, \tilde{v}) - \int_{\mathcal{K}} \tilde{p} \tilde{\mathcal{D}} \tilde{v} |\det \kappa'|^{-1} dy = \tilde{F}(\tilde{v}) \quad \text{for all } \tilde{v} \in V_{\beta,\delta}^{1,s'}(\mathcal{K})^3, \quad S_j \tilde{v} = 0 \text{ on } \Gamma_j^\circ, \quad (4.6)$$

$$-\tilde{\mathcal{D}} \tilde{u} = \tilde{g} \text{ in } \mathcal{K}, \quad S_j \tilde{u} = \tilde{h}_j \text{ on } \Gamma_j^\circ, \quad (4.7)$$

where  $\tilde{u} = u \circ \kappa^{-1}$ ,  $\tilde{F}(\tilde{v}) = F(\tilde{v} \circ \kappa)$ ,  $\Gamma_j^\circ$  are the faces of  $\mathcal{K}$ ,  $\tilde{\mathcal{D}}$  is a first order differential operator of the form

$$\tilde{\mathcal{D}} \tilde{u} = (D(y) \nabla_y) \cdot \tilde{u},$$

and  $\tilde{b}$  is a bilinear form having the representation

$$\tilde{b}(\tilde{u}, \tilde{v}) = 2\nu \int_{\mathcal{K}} \sum_{i,j=1}^3 B_{i,j}(y) \partial_{y_i} \tilde{u} \cdot \partial_{y_j} \tilde{v} dy.$$

Here  $D(y)$  and  $B_{i,j}(y)$  are quadratic matrices such that  $D(0) = I$  and

$$\sum_{i,j=1}^3 B_{i,j}(0) \partial_{y_i} \tilde{u} \cdot \partial_{y_j} \tilde{v} = \sum_{i,j=1}^3 \varepsilon_{i,j}(\tilde{u}) \varepsilon_{i,j}(\tilde{v}).$$

Let  $\zeta$  be an infinitely differentiable cut-off function on  $[0, \infty)$  equal to 1 in  $[0, 1)$  and to zero in  $(2, \infty)$ . For arbitrary positive  $\epsilon$ , we define  $\zeta_\epsilon(y) = \zeta(|y|/\epsilon)$ . Moreover, we put  $\zeta_0 = 0$  and  $\eta_\epsilon = 1 - \zeta_\epsilon$  for  $\epsilon \geq 0$ . We consider the operator

$$V_{\beta,\delta}^{1,s}(\mathcal{K})^3 \times V_{\beta,\delta}^{0,s}(\mathcal{K}) \ni (\tilde{u}, \tilde{p}) \rightarrow (\tilde{F}, \tilde{g}, \tilde{h}) \in V_{\beta,\delta}^{-1,s}(\mathcal{K}; S) \times V_{\beta,\delta}^{0,s}(\mathcal{K}) \times \prod_j V_{\beta,\delta}^{1-1/s,s}(\Gamma_j^\circ) \quad (4.8)$$

defined by

$$\begin{aligned} b_\epsilon(\tilde{u}, \tilde{v}) - \int_{\mathcal{K}} \tilde{p} (\zeta_\epsilon \tilde{D} \tilde{v} |\det \kappa'|^{-1} + \eta_\epsilon \nabla_y \cdot \tilde{v}) dy &= \tilde{F}(\tilde{v}) \quad \text{for all } \tilde{v} \in V_{\beta,\delta}^{1,s'}(\mathcal{K})^3, \quad S_j \tilde{v} = 0 \text{ on } \Gamma_j^\circ, \\ -(\zeta_\epsilon D(y) \nabla_y + \eta_\epsilon \nabla_y) \cdot \tilde{u} &= \tilde{g} \quad \text{in } \mathcal{K}, \quad S_j \tilde{u} = \tilde{h}_j \quad \text{on } \Gamma_j^\circ, \end{aligned}$$

where

$$b_\epsilon(\tilde{u}, \tilde{v}) = 2\nu \int_{\mathcal{K}} \sum_{i,j=1}^3 (\zeta_\epsilon B_{i,j}(y) + \eta_\epsilon B_{i,j}(0)) \partial_{y_i} \tilde{u} \cdot \partial_{y_j} \tilde{v} dy.$$

We denote the operator (4.8) by  $\tilde{\mathcal{A}}_\epsilon$ . According to Lemma 4.1, the operator  $\tilde{\mathcal{A}}_0$  is an isomorphism. Since the norm of  $\tilde{\mathcal{A}}_0 - \tilde{\mathcal{A}}_\epsilon$  is small for small  $\epsilon$ , the operator  $\tilde{\mathcal{A}}_\epsilon$  is an isomorphism if  $\epsilon \leq \epsilon_0$  and  $\epsilon_0$  is sufficiently small. We may assume that  $\zeta_\epsilon = 1$  on  $\mathcal{V}$  for  $\epsilon = \epsilon_0$ . Then problem (4.6), (4.7) can be written as  $\tilde{\mathcal{A}}_{\epsilon_0}(\tilde{u}, \tilde{p}) = (\tilde{F}, \tilde{g}, \tilde{h})$  if the supports of  $\tilde{u}$  and  $\tilde{p}$  are contained in  $\mathcal{V}$ . Let

$$u(x) = \tilde{u}(\kappa(x)), \quad p(x) = \tilde{p}(\kappa(x)) \quad \text{for } x \in \mathcal{U}, \quad \text{where } (\tilde{u}, \tilde{p}) = \tilde{\mathcal{A}}_{\epsilon_0}^{-1}(\tilde{F}, \tilde{g}, \tilde{h}). \quad (4.9)$$

Outside  $\mathcal{U}$ , let  $(u, p)$  be continuously extended to a vector function from  $V_{\beta,\delta}^{1,s}(\mathcal{G})^3 \times V_{\beta,\delta}^{0,s}(\mathcal{G})$ . The so defined mapping  $(F, g, h) \rightarrow (u, p)$  is denoted by  $\mathcal{R}$  and has the desired properties.

Suppose now that  $\bar{\mathcal{U}}$  contains an edge point  $\xi \in M_1$  but no points of other edges and no vertices of  $\mathcal{G}$ . Then again there exists a diffeomorphism mapping  $\mathcal{U}$  onto a subset of a cone  $\mathcal{K}$ . We assume that the point  $\kappa(\xi)$  lies on the edge  $M_1^\circ$  of  $\mathcal{K}$  and coincides with the origin (in contrast to the first part of the proof, the vertex of the cone is not the origin). Let  $\tilde{\mathcal{A}}_\epsilon$  be the same operator as above. Then there exist a number  $\beta_0$  and a tuple  $\delta', \delta'_1 = \delta_1$ , such that  $\tilde{\mathcal{A}}_0$  and for sufficiently small  $\epsilon$  also  $\tilde{\mathcal{A}}_\epsilon$  are isomorphisms

$$V_{\beta_0,\delta'}^{1,s}(\mathcal{K})^3 \times V_{\beta_0,\delta'}^{0,s}(\mathcal{K}) \rightarrow V_{\beta_0,\delta'}^{-1,s}(\mathcal{K}; S) \times V_{\beta_0,\delta'}^{0,s}(\mathcal{K}) \times \prod_j V_{\beta_0,\delta'}^{1-1/s,s}(\Gamma_j^\circ).$$

Since  $\bar{\mathcal{U}}$  does not contain points of the edges  $M_k$ ,  $k \neq 1$ , the vector function (4.9) can be continuously extended to a vector function from  $V_{\beta,\delta}^{1,s}(\mathcal{G})^3 \times V_{\beta,\delta}^{0,s}(\mathcal{G})$ . The so defined mapping  $(F, g, h) \rightarrow (u, p)$  defines the desired operator  $\mathcal{R}$ . Analogously, the lemma can be proved for the case when  $\bar{\mathcal{U}}$  contains no edge points of  $\mathcal{G}$ .  $\square$

**Remark 4.1** Suppose that

$$2 - \inf_{\xi \in M_k} \operatorname{Re} \lambda_1(\xi) < \delta'_k + 2/t < 2 + \inf_{\xi \in M_k} \operatorname{Re} \lambda_1(\xi) \quad \text{for } k = 1, \dots, m$$

and there are no eigenvalues of the pencil  $\mathfrak{A}_j(\lambda)$  in the closed strip between the lines  $\operatorname{Re} \lambda = 1 - \beta_j - 3/s$  and  $\operatorname{Re} \lambda = 2 - \beta'_j - 3/t$ . Then it follows from Lemma 4.2 that the operator  $\mathcal{R}$  constructed in the proof of Lemma 4.3 continuously maps the subspace of all  $(F, g, h)$ , where

$$g \in V_{\beta, \delta}^{0, s}(\mathcal{G}) \cap V_{\beta', \delta'}^{1, t}(\mathcal{G}), \quad h_j \in V_{\beta, \delta}^{1-1/s, s}(\Gamma_j) \cap V_{\beta', \delta'}^{2-1/t, t}(\Gamma_j)$$

and the functional  $F \in V_{\beta, \delta}^{-1, s}(\mathcal{G}; S)$  has the form

$$F(v) = \int_{\mathcal{G}} f \cdot v \, dx + \sum_j \int_{\Gamma_j} \Phi_j \cdot v \, dx \quad \text{with vector functions } f \in V_{\beta', \delta'}^{0, t}(\mathcal{G})^3, \quad \Phi_j \in V_{\beta', \delta'}^{1-1/t, t}(\Gamma_j)^3,$$

into  $V_{\beta', \delta'}^{2, t}(\mathcal{G})^3 \times V_{\beta', \delta'}^{1, t}(\mathcal{G})$ .

**Lemma 4.4** *Suppose that there are no eigenvalues of the pencil  $\mathfrak{A}_j(\lambda)$  on the line  $\operatorname{Re} \lambda = 1 - \beta_j - 3/s$ ,  $j = 1, \dots, d$ , and the components of  $\delta$  satisfy (4.5). Then there exists a continuous operator*

$$\mathcal{R} : V_{\beta, \delta}^{-1, s}(\mathcal{G}; S) \times V_{\beta, \delta}^{0, s}(\mathcal{K}) \times \prod_j V_{\beta, \delta}^{1-1/s, s}(\Gamma_j) \rightarrow V_{\beta, \delta}^{1, s}(\mathcal{G})^3 \times V_{\beta, \delta}^{0, s}(\mathcal{G})$$

such that  $\mathcal{R}\mathcal{A}_{s, \beta, \delta} - I$  and  $\mathcal{A}_{s, \beta, \delta}\mathcal{R} - I$  are compact operators in  $V_{\beta, \delta}^{1, s}(\mathcal{G})^3 \times V_{\beta, \delta}^{0, s}(\mathcal{G})$  and  $V_{\beta, \delta}^{-1, s}(\mathcal{G}; S) \times V_{\beta, \delta}^{0, s}(\mathcal{G}) \times \prod_j V_{\beta, \delta}^{1-1/s, s}(\Gamma_j)$ , respectively.

*P r o o f.* For the sake of brevity, we write  $\mathcal{A}$  instead of  $\mathcal{A}_{s, \beta, \delta}$ . Let  $\{\mathcal{U}_j\}$  be a sufficiently fine open covering of  $\mathcal{G}$ , and let  $\varphi_j, \psi_j$  be infinitely differentiable functions such that

$$\operatorname{supp} \varphi_j \subset \operatorname{supp} \psi_j \subset \mathcal{U}_j, \quad \varphi_j \psi_j = \varphi_j, \quad \text{and} \quad \sum_j \varphi_j = 1.$$

For every  $j$ , there exists an operator  $\mathcal{R}_j$  having the properties of Lemma 4.2 for  $\mathcal{U} = \mathcal{U}_j \cap \mathcal{G}$ . We consider the operator  $\mathcal{R}$  defined by

$$\mathcal{R}(F, g, h) = \sum_j \varphi_j \mathcal{R}_j \psi_j(F, g, h).$$

Obviously,

$$\mathcal{R}\mathcal{A}(u, p) = \sum_j \varphi_j \mathcal{R}_j (\mathcal{A}\psi_j(u, p) - [\mathcal{A}, \psi_j](u, p)) = (u, p) - \sum_j \varphi_j \mathcal{R}_j [\mathcal{A}, \psi_j](u, p),$$

where  $[\mathcal{A}, \psi_j]$  is the commutator of  $\mathcal{A}$  and  $\psi_j$ . Here, the mapping  $(u, p) \rightarrow [\mathcal{A}, \psi_j](u, p)$  is continuous from  $V_{\beta, \delta}^{1, s}(\mathcal{G})^3 \times V_{\beta, \delta}^{0, s}(\mathcal{G})$  into the set of all  $(F, g, h)$ , where  $g \in V_{\beta', \delta'}^{1, s}(\mathcal{G})$ ,  $h = 0$ , and  $F$  is a functional of the form

$$F(v) = \int_{\mathcal{G}} f \cdot v \, dx + \sum_j \int_{\Gamma_j} \Phi_j \cdot v \, dx, \quad \text{where } f \in V_{\beta', \delta'}^{0, s}(\mathcal{G})^3, \quad \Phi_j \in V_{\beta', \delta'}^{1-1/s, s}(\Gamma_j)^3,$$

with arbitrary  $\beta' \geq \beta$ ,  $\delta' \geq \delta$ . We can choose  $\beta'$  and  $\delta'$  such that  $\beta_j \leq \beta'_j < \beta_j + 1$  for  $j = 1, \dots, d$ ,  $\delta_k \leq \delta'_k < \delta_k + 1$  for  $k = 1, \dots, m$ , and  $\beta', \delta'$  satisfy the conditions of Remark 4.1 with  $t = s$ . Then the mapping  $(u, p) \rightarrow \mathcal{R}_j [\mathcal{A}, \psi_j](u, p)$  is continuous from  $V_{\beta, \delta}^{1, s}(\mathcal{G})^3 \times V_{\beta, \delta}^{0, s}(\mathcal{G})$  into  $V_{\beta', \delta'}^{2, s}(\mathcal{G})^3 \times V_{\beta', \delta'}^{1, s}(\mathcal{G})$ . Since the last space is compactly imbedded into  $V_{\beta, \delta}^{1, s}(\mathcal{G})^3 \times V_{\beta, \delta}^{0, s}(\mathcal{G})$  for  $\beta'_j < \beta_j + 1$ ,  $\delta'_k < \delta_k + 1$  (cf. [9, Le.6.2.1]), it follows that  $\mathcal{R}\mathcal{A} - I$  is compact. Analogously, the compactness of  $\mathcal{A}\mathcal{R} - I$  holds.  $\square$

From Lemma 4.4 it follows that the operator  $\mathcal{A}_{s, \beta, \delta}$  is Fredholm under the conditions of this lemma. Using Theorem 1.1 and Lemma 4.2, we can prove the following theorem on the existence of unique solutions.

**Theorem 4.1** Let  $F \in V_{\beta,\delta}^{-1,s}(\mathcal{G}; S)$ ,  $g \in V_{\beta,\delta}^{0,s}(\mathcal{G})$ , and  $h_j \in V_{\beta,\delta}^{1-1/s,s}(\Gamma_j)^{3-d_j}$ . We suppose that the components of  $\delta$  satisfy condition (4.3) and that there are no eigenvalues of the pencils  $\mathfrak{A}_j(\lambda)$  in the closed strip between the lines  $\operatorname{Re} \lambda = -1/2$  and  $\operatorname{Re} \lambda = 1 - \beta - 3/s$ . In the case when  $d_j \in \{0, 2\}$  for all  $j$ , we assume in addition that  $g$  and  $h_j$  satisfy condition (1.11). Then there exists a solution  $(u, p) \in V_{\beta,\delta}^{1,s}(\mathcal{G})^3 \times V_{\beta,\delta}^{0,s}(\mathcal{G})$  of problem (1.9), (1.10). (Here  $V$  has to be replaced by the set of all  $v \in V_{-\beta,-\delta}^{1,s'}(\mathcal{G})$  satisfying  $S_j v = 0$  on  $\Gamma_j$ .) The vector function  $u$  is unique,  $p$  is unique if  $d_j \notin \{0, 2\}$  for at least one  $j$ ,  $p$  is unique up to a constant if  $d_j \in \{0, 2\}$  for all  $j$ .

**P r o o f.** First note that in the case when  $d_j \in \{0, 2\}$  for all  $j$ , the spectra both of  $\mathfrak{A}_j(\lambda)$  and  $A_k(\lambda)$  contain the eigenvalue  $\lambda = 1$ . An eigenvector corresponding to this eigenvalue is  $(U, P) = (0, 1)$ . Furthermore, the spectra of the pencils  $\mathfrak{A}_j(\lambda)$  contain the eigenvalue  $\lambda = -2$ . Therefore from the conditions of the lemma on  $\beta$  and  $\delta$  it follows in particular that  $0 < \beta_j + 3/s < 3$  and  $0 < \delta_k + 2/s < 2$ . Then  $L_\infty(\mathcal{G}) \subset V_{\beta,\delta}^{0,s}(\mathcal{G}) \subset L_1(\mathcal{G})$ . In particular, any constant belongs to the space  $V_{\beta,\delta}^{0,s}(\mathcal{G})$ , and the integrals in (1.11) exist for  $g \in V_{\beta,\delta}^{0,s}(\mathcal{G})$ ,  $h_j \in V_{\beta,\delta}^{1-1/s,s}(\Gamma_j)^{3-d_j}$ .

We prove the uniqueness of  $u$  and  $p$ . Let  $(u, p) \in V_{\beta,\delta}^{1,s}(\mathcal{G})^3 \times V_{\beta,\delta}^{0,s}(\mathcal{G})$  be a solution of problem (1.9), (1.10) with  $F = 0$ ,  $g = 0$ ,  $h_j = 0$ . Then according to Lemma 4.2, we have  $u \in V_{0,0}^{1,2}(\mathcal{G})^3 \subset W^{1,2}(\mathcal{G})^3$  and  $p \in L_2(\mathcal{G})$ . From Theorem 1.1 it follows that  $u = 0$  and  $p$  is constant,  $p = 0$  if  $d_j \in \{1, 3\}$  for at least one  $j$ .

We prove the existence of solutions for the case when  $d_j \in \{0, 2\}$  for all  $j$ . Due to Lemma 4.4 and the uniqueness of the solution, every solution  $(u, p) \in V_{\beta,\delta}^{1,s}(\mathcal{G})^3 \times V_{\beta,\delta}^{0,s}(\mathcal{G})$  of problem (1.9), (1.10),  $\int_{\mathcal{G}} p \, dx = 1$ , satisfies the inequality

$$\|u\|_{V_{\beta,\delta}^{1,s}(\mathcal{G})^3} + \|p\|_{V_{\beta,\delta}^{0,s}(\mathcal{G})} \leq c \left( \|F\|_{V_{\beta,\delta}^{-1,s}(\mathcal{G}; S)} + \|g\|_{V_{\beta,\delta}^{0,s}(\mathcal{G})} + \sum_j \|h_j\|_{V_{\beta,\delta}^{1-1/s,s}(\Gamma_j)} \right) \quad (4.10)$$

with a constant  $c$  independent of  $u$  and  $p$ . Let  $F \in V_{\beta,\delta}^{-1,s}(\mathcal{G}; S)$ ,  $g \in V_{\beta,\delta}^{0,s}(\mathcal{G})$  and  $h_j \in V_{\beta,\delta}^{1-1/s,s}(\Gamma_j)^{3-d_j}$  satisfying (1.11) be given. Then there exist sequences  $\{F^{(n)}\} \subset V^* \cap V_{\beta,\delta}^{-1,s}(\mathcal{G}; S)$ ,  $\{g^{(n)}\} \subset L_2(\mathcal{G}) \cap V_{\beta,\delta}^{0,s}(\mathcal{G})$  and  $\{h_j^{(n)}\} \subset W^{1/2,2}(\Gamma_j)^{3-d_j} \cap V_{\beta,\delta}^{1-1/s,s}(\Gamma_j)^{3-d_j}$  converging to  $F$ ,  $g$ , and  $h_j$ , respectively. From (1.11) it follows that the sequences of the numbers

$$a_n = \int_{\mathcal{G}} g^{(n)} \, dx + \sum_{j: d_j=0} \int_{\Gamma_j} h_j^{(n)} \cdot n \, dx + \sum_{j: d_j=2} \int_{\Gamma_j} h_j^{(n)} \, dx$$

converges to zero. Therefore, the sequence of the functions  $\tilde{g}^{(n)} = g^{(n)} - \frac{1}{|\mathcal{G}|} a_n$  converges also to  $g$ . Moreover,  $\tilde{g}^{(n)}$  and  $h_j^{(n)}$  satisfy condition (1.11). By Theorem 1.1, there exist solutions  $(u^{(n)}, p^{(n)}) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$  of problem (1.9), (1.10) with right-hand sides  $F^{(n)}$ ,  $\tilde{g}^{(n)}$  and  $h_j^{(n)}$ ,  $\int_{\mathcal{G}} p^{(n)} \, dx = 0$ . From Lemma 4.2 and from the imbedding  $W^{1,2}(\mathcal{G}) \subset V_{0,\varepsilon}^{1,2}(\mathcal{G})$  with arbitrary positive  $\varepsilon$  it follows that  $(u^{(n)}, p^{(n)}) \in V_{\beta,\delta}^{1,s}(\mathcal{G})^3 \times V_{\beta,\delta}^{0,s}(\mathcal{G})$ . Due to (4.10), the vector functions  $(u^{(n)}, p^{(n)})$  form a Cauchy sequence in  $V_{\beta,\delta}^{1,s}(\mathcal{G})^3 \times V_{\beta,\delta}^{0,s}(\mathcal{G})$ . The limit of this sequence solves problem (1.9), (1.10). The proof of the existence of solutions in the case when  $d_j \in \{1, 3\}$  for at least one  $j$  proceeds analogously.  $\square$

### 4.3 Solvability of the nonlinear problem

We consider the nonlinear problem (1.2), (1.3). For the proof of the existence of solutions in  $V_{\beta,\delta}^{1,s}(\mathcal{G})^3 \times V_{\beta,\delta}^{0,s}(\mathcal{G})$ , we have to show inequalities analogous to (1.13), (1.14) for the operator

$$V_{\beta,\delta}^{1,s}(\mathcal{G})^3 \ni u \rightarrow Qu = (u \cdot \nabla)u \in V_{\beta,\delta}^{-1,s}(\mathcal{G}; S).$$

**Lemma 4.5** Suppose that  $s > 3/2$ ,  $\beta_j + 3/s \leq 2$  for  $j = 1, \dots, N$ , and  $\delta_k + 3/s \leq 2$  for  $k = 1, \dots, m$ . Then

$$\|u \partial_{x_j} v\|_{V_{\beta,\delta}^{-1,s}(\mathcal{G})^3} \leq c \|u\|_{V_{\beta,\delta}^{1,s}(\mathcal{G})} \|v\|_{V_{\beta,\delta}^{1,s}(\mathcal{G})}$$

for all  $u, v \in V_{\beta,\delta}^{1,s}(\mathcal{G})$ ,  $j = 1, 2, 3$ .

P r o o f. Let  $t$  be a real number such that

$$t \geq 3, \quad t \geq s, \quad \text{and} \quad \frac{3}{s} - 1 \leq \frac{3}{t} < 3 - \frac{3}{s}.$$

For example, we can put  $t = \max(s, 3)$ . Then according to Lemma 2.6, the space  $V_{\beta, \delta}^{1, s}(\mathcal{G})$  is continuously imbedded into  $V_{\beta', \delta'}^{0, t}(\mathcal{G})$ , where  $\beta'_j = \beta - 1 + 3s^{-1} - 3t^{-1}$ ,  $\delta'_k = \delta_k - 1 + 3s^{-1} - 3t^{-1}$ . Let  $q^{-1} = s^{-1} + t^{-1}$ . From the conditions on  $t$  it follows that  $q > 1$ . By Hölders's inequality, we have

$$\|u \partial_{x_j} v\|_{V_{\beta+\beta', \delta+\delta'}^{0, q}(\mathcal{G})} \leq c \|u\|_{V_{\beta', \delta'}^{0, t}(\mathcal{G})} \|\partial_{x_j} v\|_{V_{\beta, \delta}^{0, s}(\mathcal{G})}$$

for arbitrary  $u, v \in V_{\beta, \delta}^{1, s}(\mathcal{G})$ . Using the continuity of the imbedding  $V_{\beta+\beta', \delta+\delta'}^{0, q}(\mathcal{G}) \subset V_{\beta, \delta}^{-1, s}(\mathcal{G})$  which follows from Corollary 2.2, we obtain the desired inequality.  $\square$

As a consequence of Lemma 4.5, the following inequalities hold for arbitrary  $u, v \in V_{\beta, \delta}^{1, s}(\mathcal{G})^3$ :

$$\|Qu\|_{V_{\beta, \delta}^{-1, s}(\mathcal{G}; S)} \leq c \|u\|_{V_{\beta, \delta}^{1, s}(\mathcal{G})^3}^2, \quad (4.11)$$

$$\|Qu - Qv\|_{V_{\beta, \delta}^{-1, s}(\mathcal{G}; S)} \leq c (\|u\|_{V_{\beta, \delta}^{1, s}(\mathcal{G})^3} + \|v\|_{V_{\beta, \delta}^{1, s}(\mathcal{G})^3}) \|u - v\|_{V_{\beta, \delta}^{1, s}(\mathcal{G})^3}. \quad (4.12)$$

Using these estimates, the following theorem can be proved analogously to Theorem 1.2.

**Theorem 4.2** *Let the conditions of Theorem 4.1 on  $F$ ,  $g$ ,  $h_j$ ,  $\beta$  and  $\delta$  be satisfied. In addition, we assume that  $s > 3/2$ ,  $\beta_j + 3/s \leq 2$ ,  $\delta_k + 3/s \leq 2$  and that*

$$\|F\|_{V_{\beta, \delta}^{-1, s}(\mathcal{G}; S)} + \|g\|_{V_{\beta, \delta}^{0, s}(\mathcal{G})} + \sum_j \|h_j\|_{V_{\beta, \delta}^{1-1/s, s}(\Gamma_j)}^{3-d_j}$$

*is sufficiently small. Then there exists a solution  $(u, p) \in V_{\beta, \delta}^{1, s}(\mathcal{G})^3 \times V_{\beta, \delta}^{0, s}(\mathcal{G})$  of problem (1.4), (1.5), where  $V$  has to be replaced by the set of all  $v \in V_{-\beta, -\delta}^{1, s'}(\mathcal{G})^3$  such that  $S_j v = 0$  on  $\Gamma_j$ ,  $j = 1, \dots, N$ . The function  $u$  is unique on the set of all functions with norm less than a certain positive  $\varepsilon$ ,  $p$  is unique if  $d_j \in \{1, 3\}$  for at least one  $j$ , otherwise  $p$  is unique up to a constant.*

P r o o f. Let  $(u^{(0)}, p^{(0)}) \in V_{\beta, \delta}^{1, s}(\mathcal{G})^3 \times V_{\beta, \delta}^{0, s}(\mathcal{G})$ . By Theorem 4.1, the norm of  $u^{(0)}$  can be assumed to be small. Obviously,  $(u, p)$  is a solution of problem (1.4), (1.5) if and only if  $(w, q) = (u - u^{(0)}, p - p^{(0)})$  is a solution of the problem

$$\begin{aligned} b(w, v) - \int_{\mathcal{G}} q \nabla \cdot v \, dx &= - \int_{\mathcal{G}} Q(w + u^{(0)}) \cdot v \, dx \quad \text{for all } v \in V_{-\beta, -\delta}^{1, s'}(\mathcal{G})^3, \quad S_j v|_{\Gamma_j} = 0, \\ \nabla \cdot w &= 0 \quad \text{in } \mathcal{G}, \quad S_j w = 0 \quad \text{on } \Gamma_j, \quad j = 1, \dots, N. \end{aligned}$$

This problem can be written as

$$(w, q) = -AQ(w + u^{(0)})$$

where  $A$  is the inverse to the operator  $F \rightarrow \mathcal{A}_{s, \beta, \delta}(F, 0, 0)$  considered in the preceding subsection. Due to (4.12), the operator  $(w, q) \rightarrow -AQ(w + u^{(0)})$  is contractive on the set of all  $(w, q) \in V_{\beta, \delta}^{1, s}(\mathcal{G})^3 \times V_{\beta, \delta}^{0, s}(\mathcal{G})$  with norm  $\leq \varepsilon$  if  $\varepsilon$  and the norm of  $u^{(0)}$  are sufficiently small. Hence this operator has a fixed point. This proves the theorem.  $\square$

Finally, we give a result in nonweighted Sobolev spaces which follows immediately from the last theorem.

*Let  $\mathcal{G}$  be a polyhedron. We assume that on every face  $\Gamma_j$ , one of the boundary conditions (i)–(iii) is given, that the Dirichlet condition is given on at least one of the adjoining faces of every edge  $M_k$ ,*

and that  $\theta_k \leq 3\pi/2$  if the boundary conditions (ii) or (iii) are given on one of the adjoining faces of  $M_k$ . Here,  $\theta_k$  denotes again the angle at the edge  $M_k$ . Then the problem

$$\begin{aligned} b(u, v) + \int_{\mathcal{G}} \sum_{j=1}^3 u_j \frac{\partial u}{\partial x_j} \cdot v \, dx - \int_{\mathcal{G}} p \nabla \cdot v \, dx &= F(v) \quad \text{for all } v \in W^{1,s'}(\mathcal{G})^3, \quad S_j v = 0 \text{ on } \Gamma_j, \\ -\nabla \cdot u &= 0 \quad \text{in } \mathcal{G}, \quad S_j u = 0 \text{ on } \Gamma_j, \quad j = 1, \dots, N, \end{aligned}$$

with  $F \in W^{-1,s}(\mathcal{G}; S)$ ,  $3/2 < s < 3$ , has a solution  $(u, p) \in W^{1,s}(\mathcal{G})^3 \times L_s(\mathcal{G})$  if the norm of  $F$  is sufficiently small.

For the proof it suffices to note that the strip  $-1 \leq \operatorname{Re} \lambda \leq 0$  does not contain eigenvalues of the pencils  $\mathfrak{A}_j(\lambda)$  and that  $\operatorname{Re} \lambda_1^{(k)} \geq 1/3$  for  $k = 1, \dots, m$ . Therefore, the conditions of Theorem 4.2 are satisfied for  $\beta = 0$ ,  $\delta = 0$ ,  $3/2 < s < 3$ .

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